

Chapter Two: Wave Nature of Light

Light waves in a homogeneous medium

Plane electromagnetic waves

Maxwell's wave equation and diverging waves

Refractive index

Group velocity and group index

Snell's law and total internal reflection (TIR)

Fresnel's equations

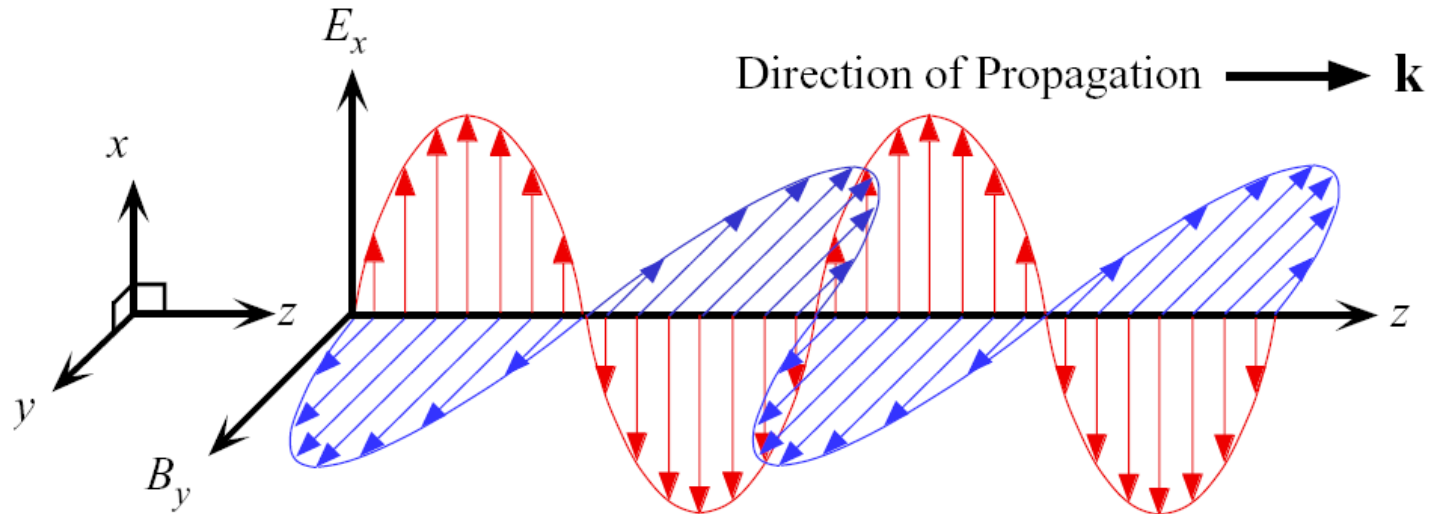
Goos-Hänchen shift and optical tunneling

Diffraction

Fraunhofer diffraction

Diffraction grating

Plane Electromagnetic Wave



An electromagnetic wave traveling along z has time-varying electric and magnetic fields that are perpendicular to each other.

$$E_x = E_0 \cos(\omega t - kz + \phi_0)$$

$$E_x(z, t) = \mathbf{Re}[E_0 \exp(j\phi_0) \exp j(\omega t - kz)]$$

$$k = 2\pi / \lambda$$

$$\omega = 2\pi\nu$$

$$\phi = \omega t - kz + \phi_0$$

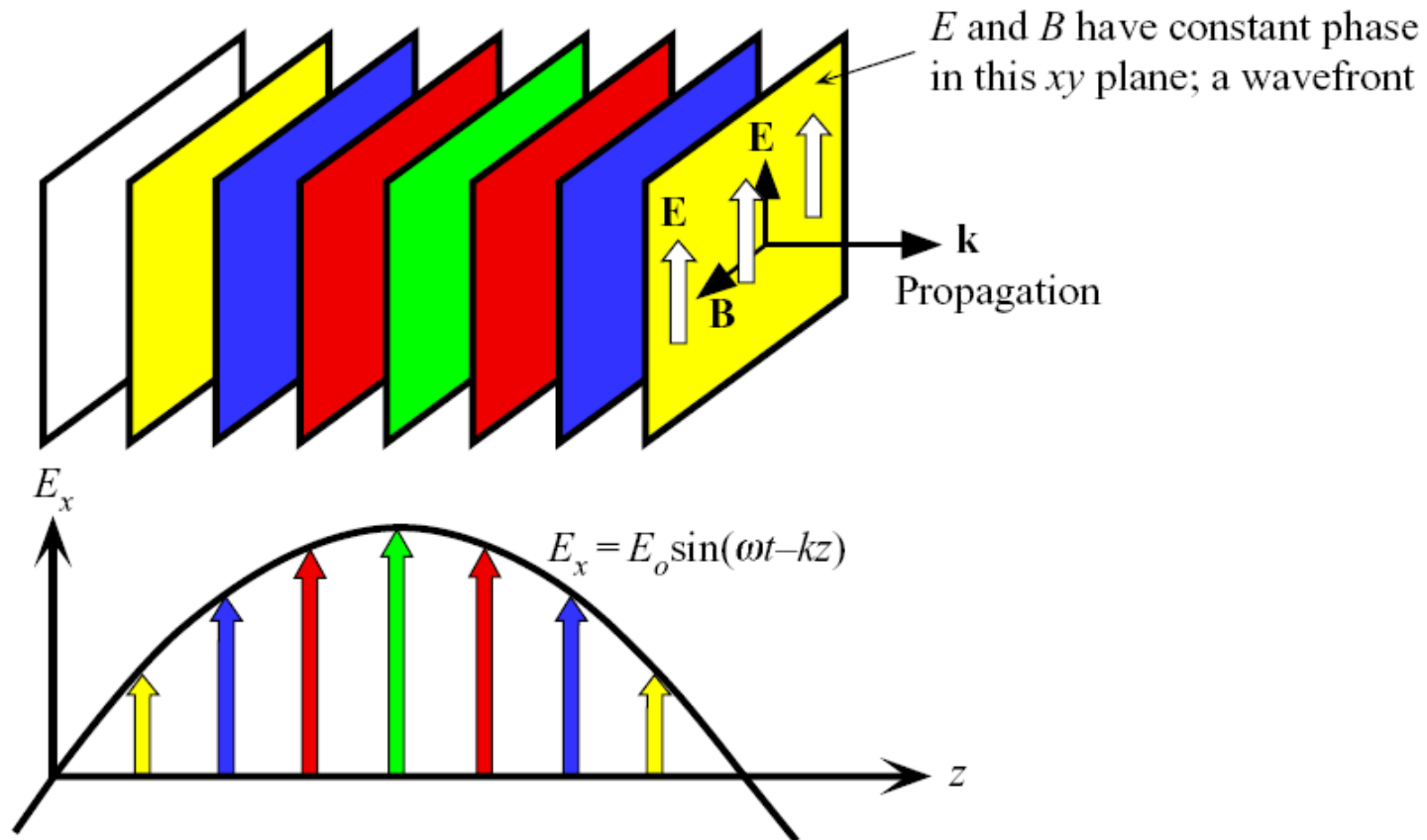
propagation constant, or wave number

angular frequency

phase

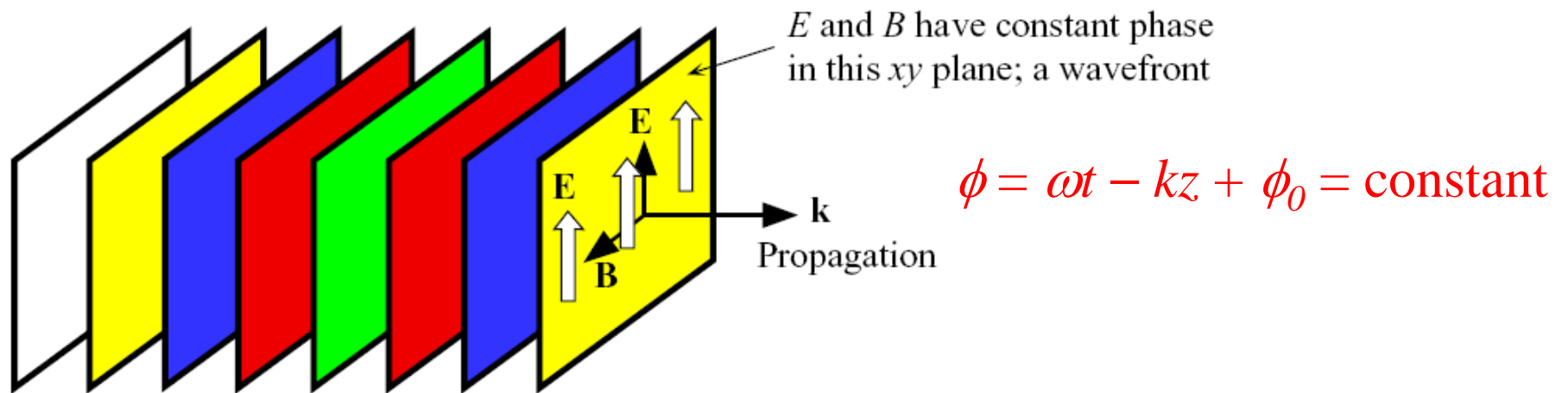
The **optical field** generally refers to the electric field.

Plane Electromagnetic Wave



A surface over which the phase of a wave is constant is referred to as a **wavefront**.

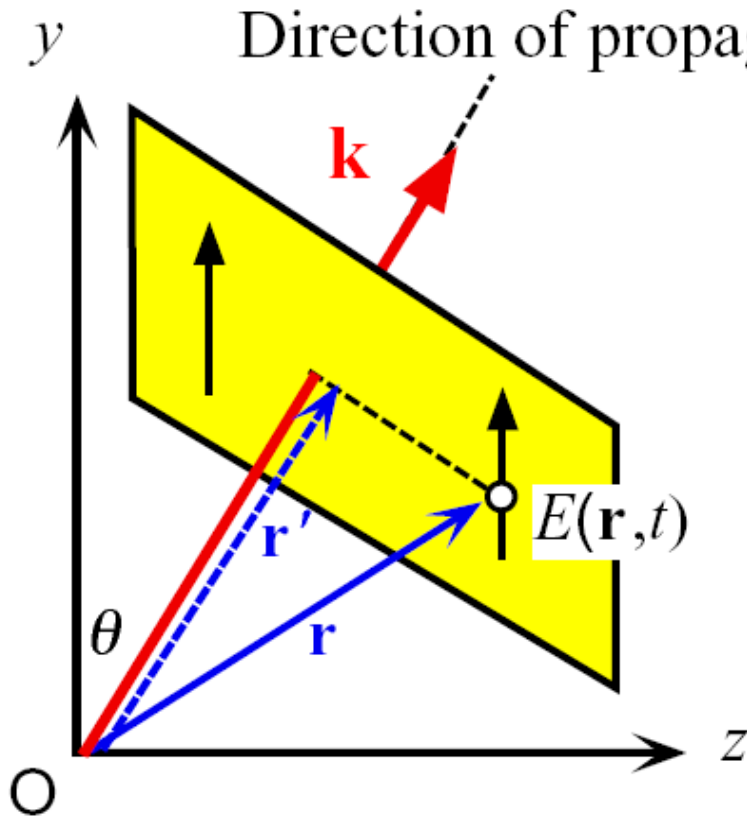
Phase Velocity



During a time interval δt , the wavefronts of constant phases move a distance δz . The **phase velocity** v is therefore

$$v = \frac{\delta z}{\delta t} = \frac{\omega}{k} = v\lambda$$

Plane Electromagnetic Wave



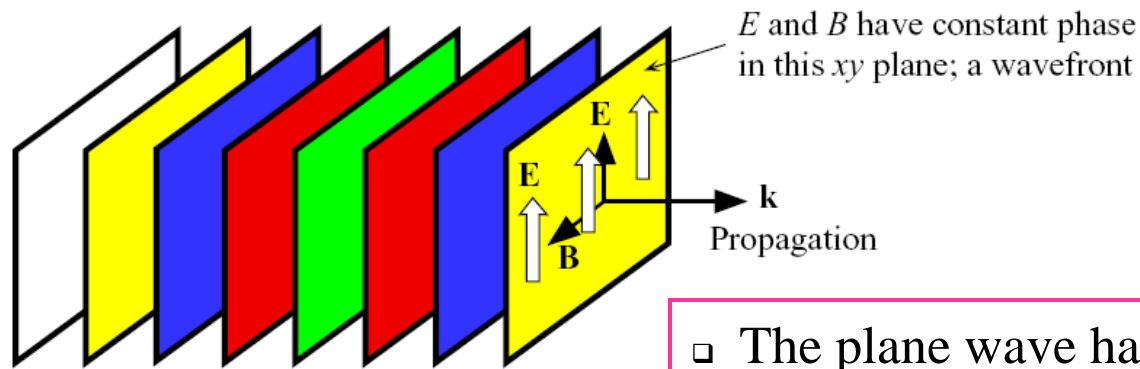
At a given time, the phase difference between two points separated by $\Delta\mathbf{r}$ is simply $\mathbf{k}\cdot\Delta\mathbf{r}$. If this phase difference is 0 or multiples of 2π , then the two points are **in phase**.

$$E(\mathbf{r}, t) = E_0 \cos(\omega t - \mathbf{k}\cdot\mathbf{r} + \phi_0)$$

$$\mathbf{k}\cdot\mathbf{r} = k_x x + k_y y + k_z z$$

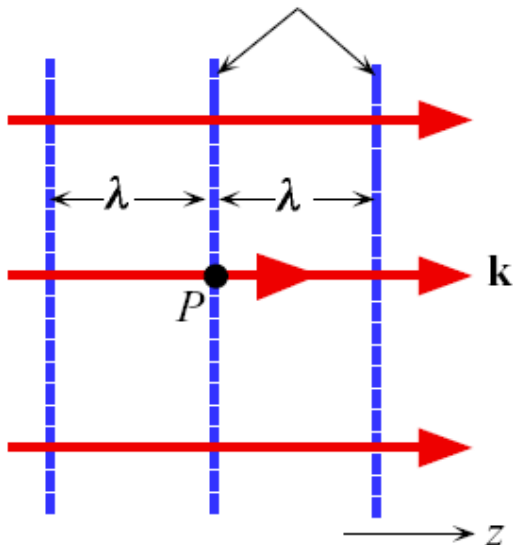
k is called the **wave vector**.

Plane Waves



$$E_x = E_0 \cos(\omega t - kz + \phi_0)$$

Wave fronts
(constant phase surfaces)



A perfect plane wave

- The plane wave has no angular separation (**no divergence**) in its wavevectors.
- E_0 does not depend on the distance from a reference point, and it is the same at all points on a given plane perpendicular to \mathbf{k} .
- The plane wave is an idealization that is useful in analyzing many wave phenomena.
- Real light beams would have finite cross sections.
- The plane wave obeys **Maxwell's EM wave equation**.

Maxwell's EM Wave Equation

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \epsilon_0 \epsilon_r \mu_0 \frac{\partial^2 E}{\partial t^2}$$

For a perfect plane wave: $E_x = E_0 \cos(\omega t - kz + \phi_0)$

Left side:

$$\frac{\partial^2 E}{\partial x^2} = 0$$

$$\frac{\partial^2 E}{\partial y^2} = 0$$

$$\begin{aligned} \frac{\partial^2 E}{\partial z^2} &= -E_0 (-k)^2 \cos(\omega t - kz + \phi_0) \\ &= -k^2 E_0 \cos(\omega t - kz + \phi_0) \end{aligned}$$

Right side:

$$\frac{\partial^2 E}{\partial t^2} = -\omega^2 E_0 \cos(\omega t - kz + \phi_0)$$

From EM:

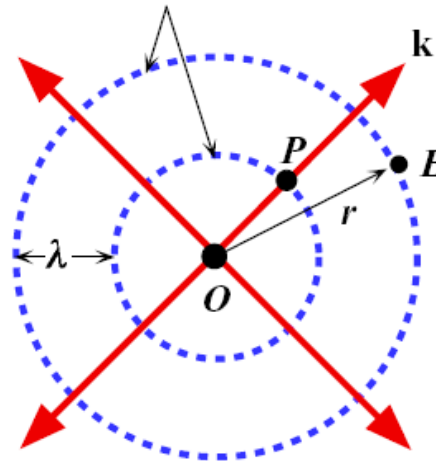
$$\begin{aligned} k^2 &= \epsilon_0 \epsilon_r \mu_0 \omega^2 \\ v = \lambda \nu &= \frac{1}{\sqrt{\epsilon_0 \epsilon_r \mu_0}} \end{aligned}$$

Diverging Waves

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \epsilon_0 \epsilon_r \mu_0 \frac{\partial^2 E}{\partial t^2}$$

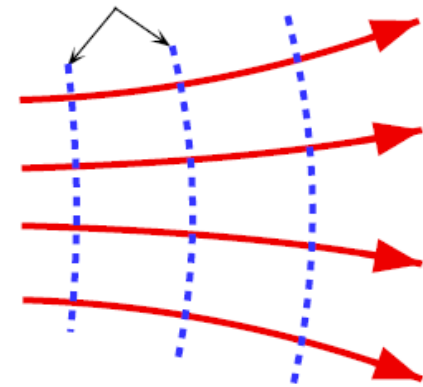
$$E = (A/r) \cos(\omega t - kr)$$

Wave fronts



A perfect spherical wave

Wave fronts

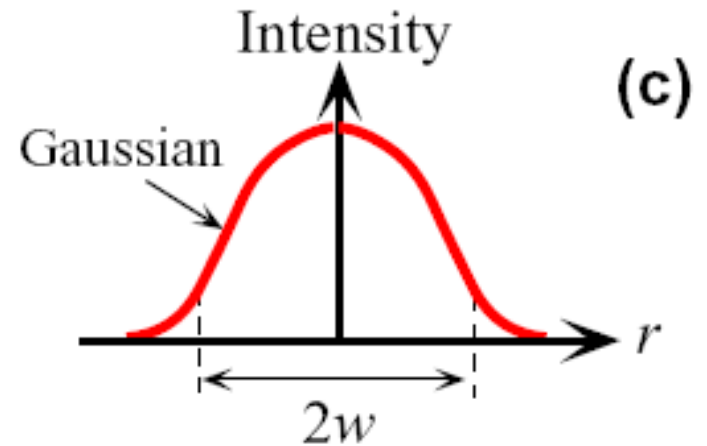
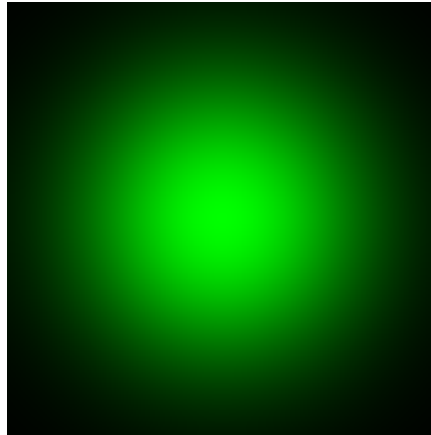


A divergent beam

- A **spherical wave** is described by a traveling wave that emerges from a point EM source and whose amplitude decays with distance r from the source.
- **Optical divergence** refers to the angular separation of wavevectors on a given wavefront.
- Plane and spherical waves represent two extremes of wave propagation behavior from perfectly parallel to fully diverging wavevectors. They are produced by two extreme sizes of EM wave sources: an infinitely large source for the plane wave and a point source for the spherical wave. A real EM source would have a finite size and finite power.

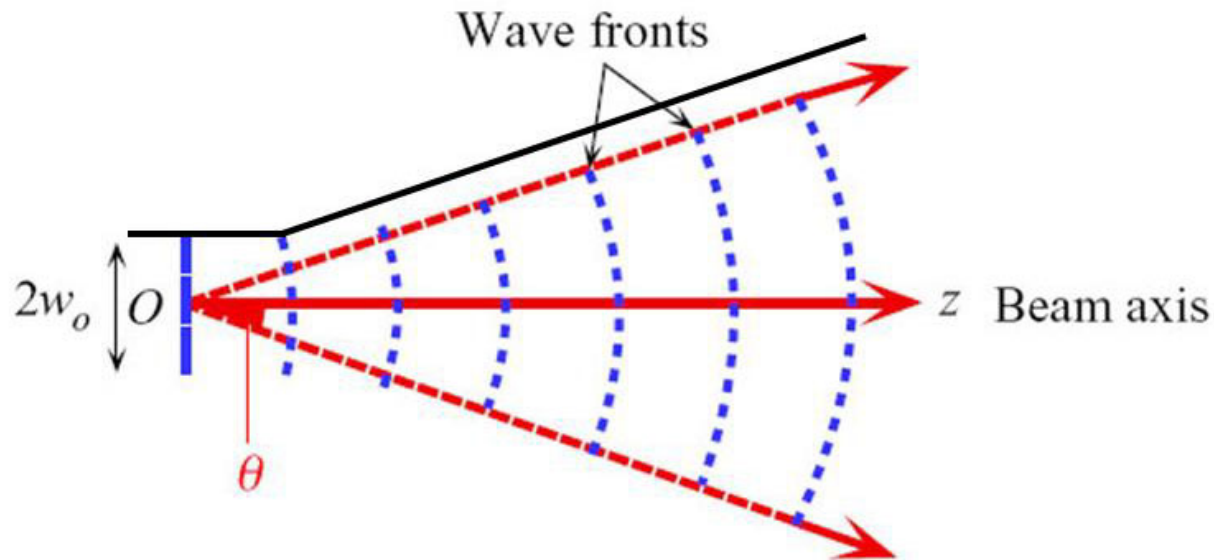
Gaussian Beams

Many laser beams can be described by Gaussian beams.



Gaussian beam has an $\exp[j(\omega t - kz)]$ dependence to describe propagation characteristics but the amplitude varies spatially away from the beam axis. The beam diameter $2w$ at any point z is defined as the diameter at which the beam intensity has fallen to $1/e^2$ (13.5%) of its maximum.

Gaussian Beams



Diffraction causes light waves to spread transversely as they propagate, and it is therefore impossible to have a perfectly collimated beam. The spreading of a laser beam is in precise accord with the predictions of pure diffraction theory. Even if a Gaussian laser beam wavefront were made perfectly flat at some plane, it would quickly acquire curvature and begin spreading out. The finite width $2w_0$ where the wavefronts are parallel is called the **waist** of the beam. w_0 is the **waist radius** and $2w_0$ is the **spot size**. The increase in beam diameter $2w$ with z makes an angle of 2θ , which is called the **beam divergence**. Spot size and beam divergence are two important parameters when choosing lasers.

Chapter Two: Wave Nature of Light

Light waves in a homogeneous medium

Plane electromagnetic waves

Maxwell's wave equation and diverging waves

Refractive index

Group velocity and group index

Snell's law and total internal reflection (TIR)

Fresnel's equations

Goos-Hänchen shift and optical tunneling

Diffraction

Fraunhofer diffraction

Diffraction grating

Refractive Index

When an EM wave is traveling in a dielectric medium, the oscillating electric field polarizes the molecules of the medium at the frequency of the wave. The field and the induced molecular dipoles become coupled. The net effect is that the polarization mechanism delays the propagation of the EM wave. The stronger the interaction between the field and the dipoles, the slower the propagation of the wave. The relative permittivity ϵ_r measures the ease with which the medium becomes polarized and indicates the extent of interaction between the field and the induced dipoles.

In vacuum:

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

The phase velocity in a medium is:

$$v = \frac{1}{\sqrt{\epsilon_0 \epsilon_r \mu_0}}$$

The refractive index of the medium is:

$$n = \frac{c}{v} = \sqrt{\epsilon_r}$$

Refractive Index

$$k_{\text{medium}} = nk$$

$$\lambda_{\text{medium}} = \lambda/n$$

The frequency ν remains the same in different mediums.

- In non-crystalline materials (glasses and liquids), the material structure is the same in all directions and n does not depend on the direction. The refractive index is **isotropic**.
- Crystals, in general, have **anisotropic** properties. The refractive index seen by a propagating EM wave will depend on the direction of the electric field relative to crystal structures, which is further determined by both the propagation direction and the electric field oscillation direction (**polarization**).

Relative Permittivity and Refractive Index

The relative permittivity for many materials can be vastly different at high and low frequencies because different polarization mechanisms operate at these frequencies. At low frequencies, all polarization mechanisms (dipolar, ionic, electronic) contribute to ϵ_r , whereas at high (optical) frequencies only the electronic polarization can respond to the oscillating field.

Low frequency (LF) relative permittivity $\epsilon_r(\text{LF})$ and refractive index n

Material	$\epsilon_r(\text{LF})$	$\sqrt{\epsilon_r(\text{LF})}$	n (optical)	Comment
Si	11.9	3.44	3.45 (at 2.15 μm)	Electronic polarization up to optical frequencies
Diamond	5.7	2.39	2.41 (at 590 nm)	Electronic polarization up to UV light
GaAs	13.1	3.62	3.30 (at 5 μm)	Ionic polarization contributes to $\epsilon_r(\text{LF})$
SiO ₂	3.84	2.00	1.46 (at 600 nm)	Ionic polarization contributes to $\epsilon_r(\text{LF})$
Water	80	8.9	1.33 (at 600 nm)	Dipolar polarization contributes to $\epsilon_r(\text{LF})$, which is large

Chapter Two: Wave Nature of Light

Light waves in a homogeneous medium

- Plane electromagnetic waves

- Maxwell's wave equation and diverging waves

Refractive index

Group velocity and group index

Snell's law and total internal reflection (TIR)

Fresnel's equations

Goos-Hänchen shift and optical tunneling

Diffraction

- Fraunhofer diffraction

- Diffraction grating

Group Velocity

Since there are no perfect monochromatic waves in practice, we have to consider the way in which a group of waves differing slightly in wavelength travel along the same direction. For two harmonic waves of frequencies $\omega - \delta\omega$ and $\omega + \delta\omega$ and wavevectors $k - \delta k$ and $k + \delta k$ interfering with each other and traveling along the z direction:

$$E_{x,1}(z, t) = E_0 \cos[(\omega - \delta\omega)t - (k - \delta k)z]$$

$$E_{x,2}(z, t) = E_0 \cos[(\omega + \delta\omega)t - (k + \delta k)z]$$

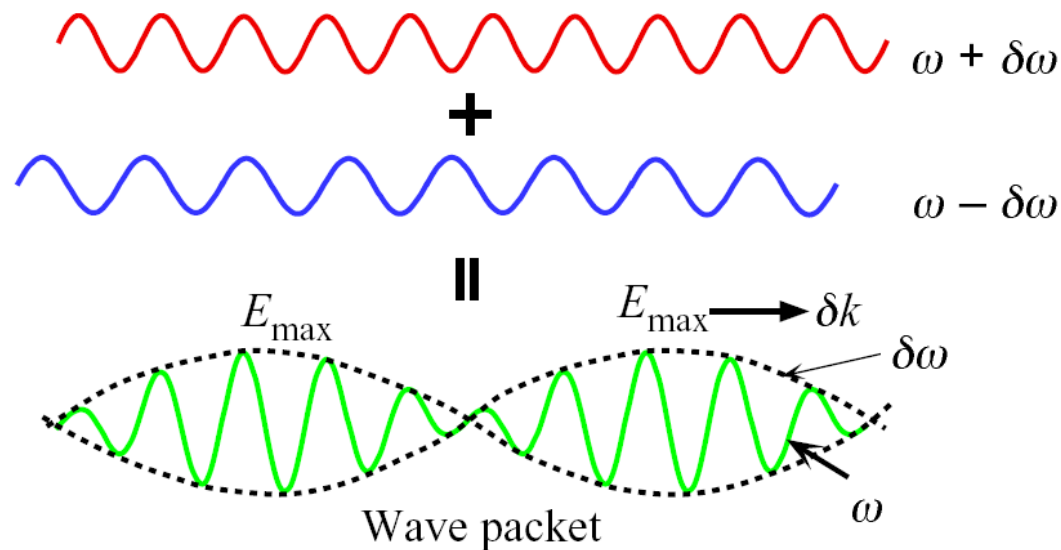
Using

$$\cos A + \cos B = 2 \cos\left[\frac{1}{2}(A - B)\right] \cos\left[\frac{1}{2}(A + B)\right]$$

Group Velocity


We obtain:

$$E_x = E_{x,1} + E_{x,2} = 2E_0 \cos[(\delta\omega)t - (\delta k)z] \cos(\omega t - kz)$$



Generate a wave packet containing an oscillating field at the mean frequency ω with the amplitude modulated by a slowly varying field of frequency $\delta\omega$.

Group Velocity

$$E_x = 2E_0 \cos[\underbrace{(\delta\omega)t - (\delta k)z}] \cos(\underbrace{\omega t - kz})$$


The maximum amplitude moves with a wavevector δk and a frequency $\delta\omega$.

$$\text{Group velocity } v_g = \frac{d\omega}{dk}$$

$$\text{Phase velocity } v = \frac{\omega}{k} = \frac{c}{n}$$

In vacuum:

$$\omega = ck$$

$$v_g = \frac{d\omega}{dk} = c$$

group velocity



phase velocity

The group velocity defines the speed with which energy or information is propagated since it defines the speed of the envelope of the amplitude variation.

In a medium with n as a function of λ :

$$\omega = ck / n$$

$$k = n\omega / c$$

$$\frac{cdk}{d\omega} = \frac{d(n\omega)}{d\omega} = n + \omega \frac{dn}{d\omega}$$

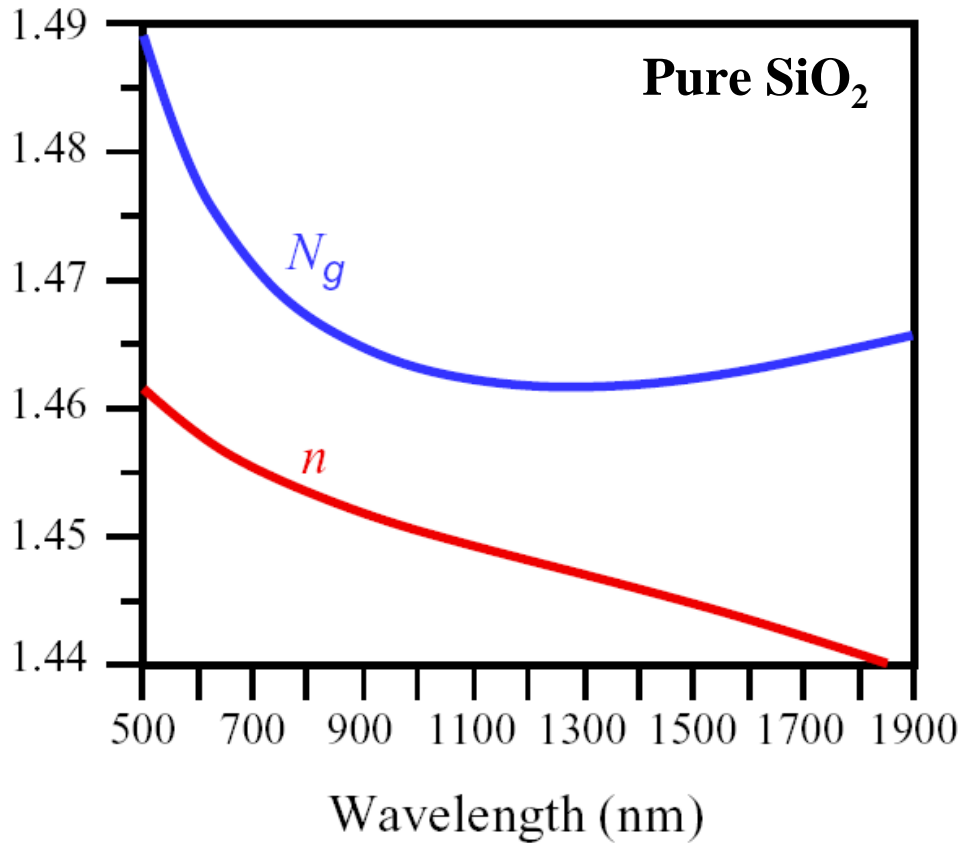
$$\lambda = \frac{c/v}{n} = \frac{2\pi c}{\omega n}$$

$$\omega \frac{dn}{d\omega} = \omega \frac{dn}{d\lambda} \frac{d\lambda}{d\omega} = -\omega \frac{dn}{d\lambda} \frac{2\pi c}{\omega^2 n} = -\lambda \frac{dn}{d\lambda}$$

$$v_g = \frac{d\omega}{dk} = \frac{c}{n - \lambda \frac{dn}{d\lambda}} = \frac{c}{N_g} \quad N_g = n - \lambda \frac{dn}{d\lambda}$$

N_g is the group index of the medium.

Group Index of the Medium



For silica, N_g is minimum around 1300 nm, which means that light waves with wavelengths close to 1300 nm travel with the same group velocity and do not experience dispersion.

Both n and N_g are functions of the free space wavelength. Such a medium is called a **dispersive medium**. n and N_g of SiO_2 are important parameters for optical fiber design in optical communications.

Chapter Two: Wave Nature of Light

Light waves in a homogeneous medium

Plane electromagnetic waves

Maxwell's wave equation and diverging waves

Refractive index

Group velocity and group index

Snell's law and total internal reflection (TIR)

Fresnel's equations

Goos-Hänchen shift and optical tunneling

Diffraction

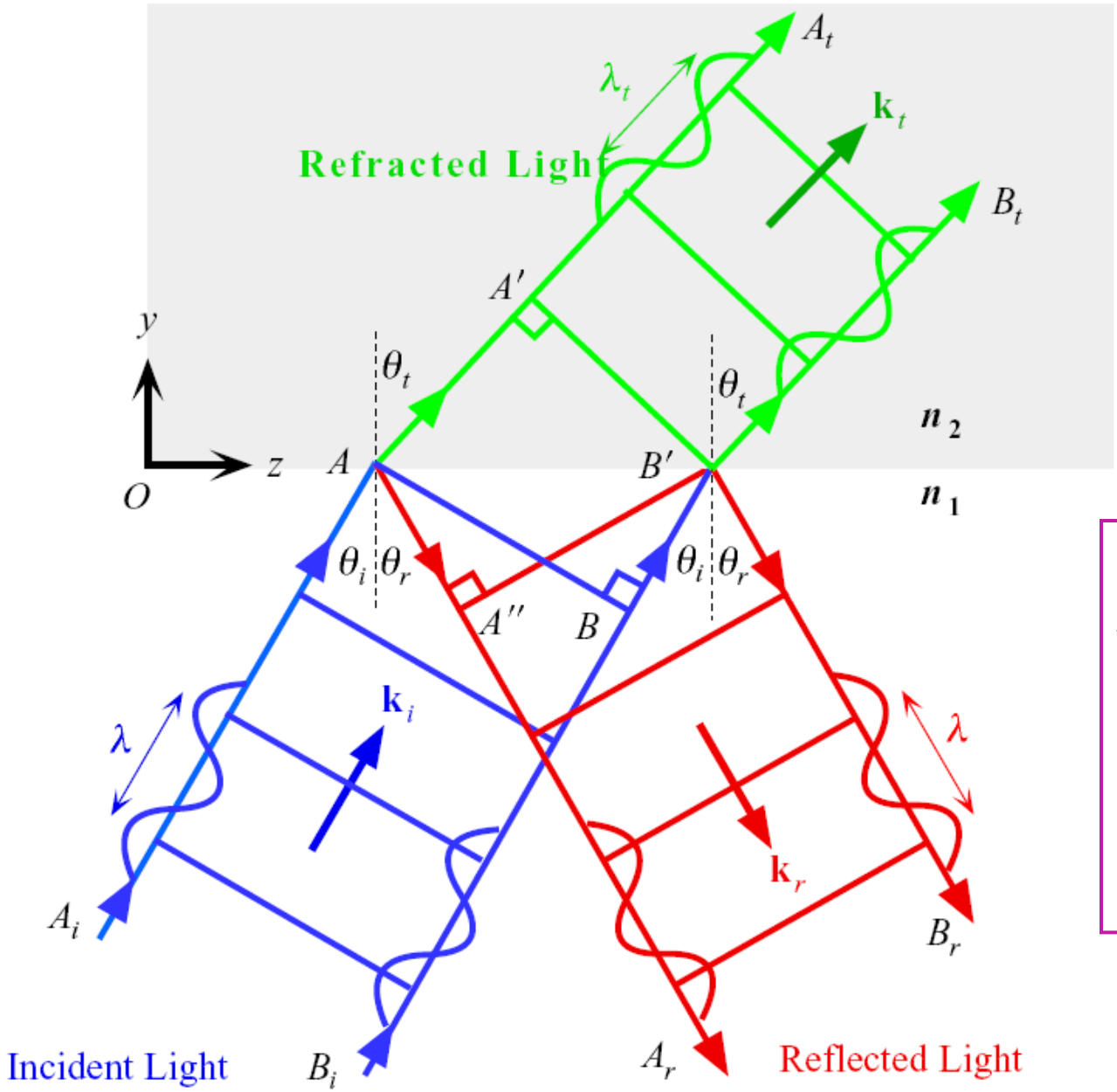
Fraunhofer diffraction

Diffraction grating

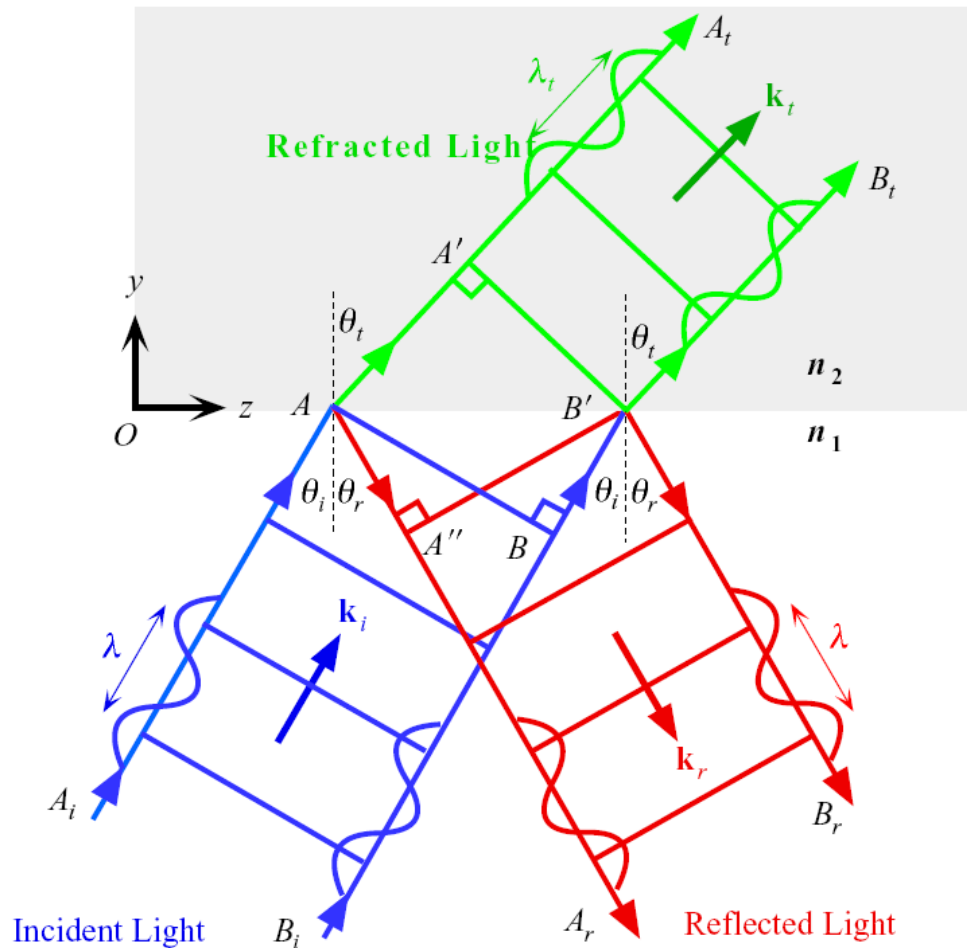
Snell's Law



Willebrord van Roijen Snell
(1580-1626)



Incident light waves A_i and B_i are in phase. Reflected waves A_r and B_r and refracted waves A_t and B_t should also be in phase, respectively.



Reflection

$$AB' = \frac{v_1 t}{\sin \theta_i} = \frac{v_1 t}{\sin \theta_r}$$

$$\theta_i = \theta_r$$

Refraction

$$AB' = \frac{v_1 t}{\sin \theta_i} = \frac{v_2 t}{\sin \theta_t}$$

$$\frac{\sin \theta_i}{\sin \theta_t} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

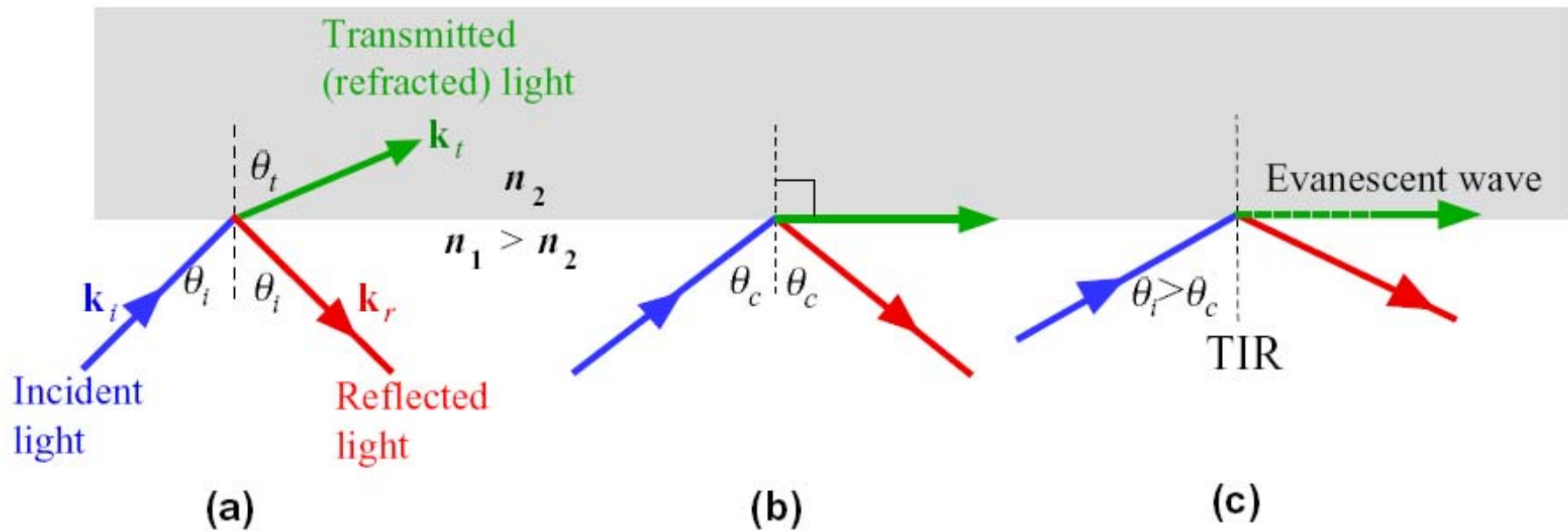
Refraction

$$\frac{\sin \theta_i}{\sin \theta_t} = \frac{n_2}{n_1}$$

Total Internal Reflection

When $n_1 > n_2$ and $\theta_t = 90^\circ$, the incidence angle is called the **critical angle** θ_c :

$$\sin \theta_c = \frac{n_2}{n_1}$$



Chapter Two: Wave Nature of Light

Light waves in a homogeneous medium

Plane electromagnetic waves

Maxwell's wave equation and diverging waves

Refractive index

Group velocity and group index

Snell's law and total internal reflection (TIR)

Fresnel's equations

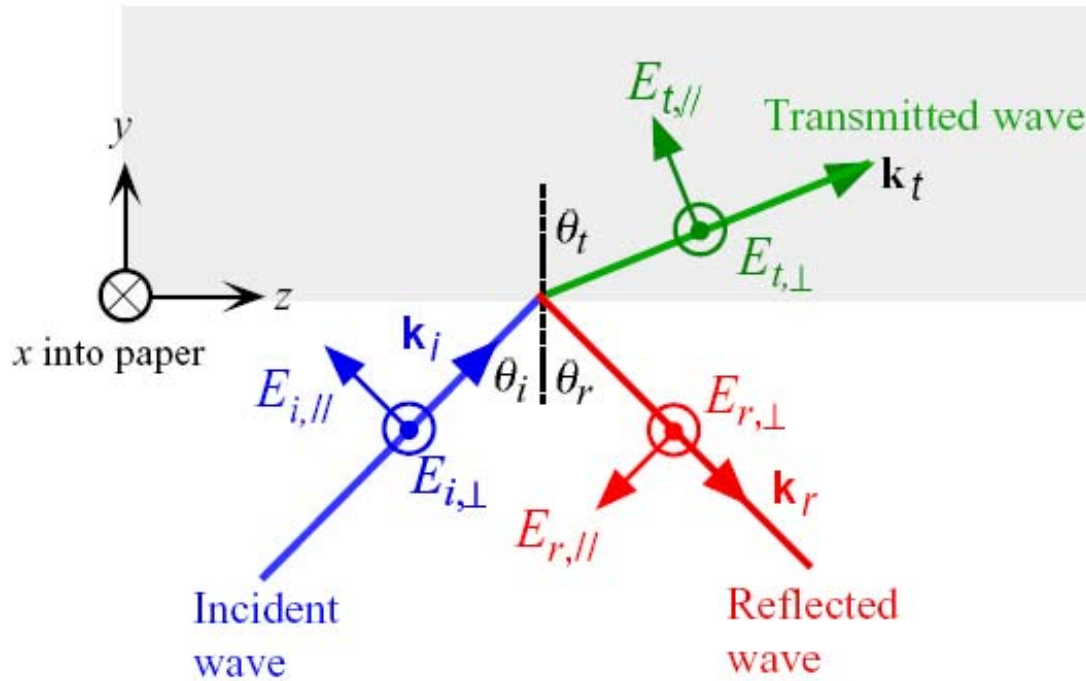
Goos-Hänchen shift and optical tunneling

Diffraction

Fraunhofer diffraction

Diffraction grating

Fresnel's Equations



The **incidence plane** is the plane containing the incident and reflected light rays.



Augustin Fresnel
(1788-1827)

Transverse electric field (TE) waves: $E_{i,\perp}$, $E_{r,\perp}$, and $E_{t,\perp}$.

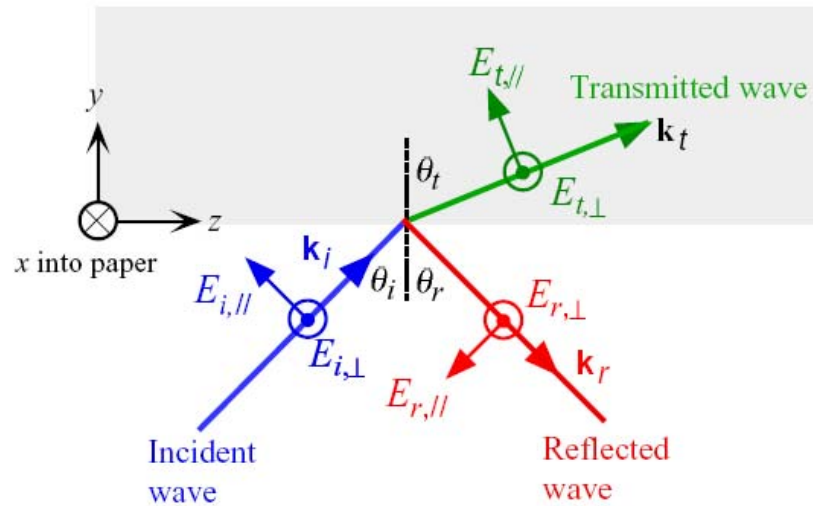
Transverse magnetic field (TM) waves: $E_{i,||}$, $E_{r,||}$, and $E_{t,||}$.

$$E_i = E_{i0} \exp j(\omega t - \mathbf{k}_i \cdot \mathbf{r})$$

$$E_r = E_{r0} \exp j(\omega t - \mathbf{k}_r \cdot \mathbf{r})$$

$$E_t = E_{t0} \exp j(\omega t - \mathbf{k}_t \cdot \mathbf{r})$$

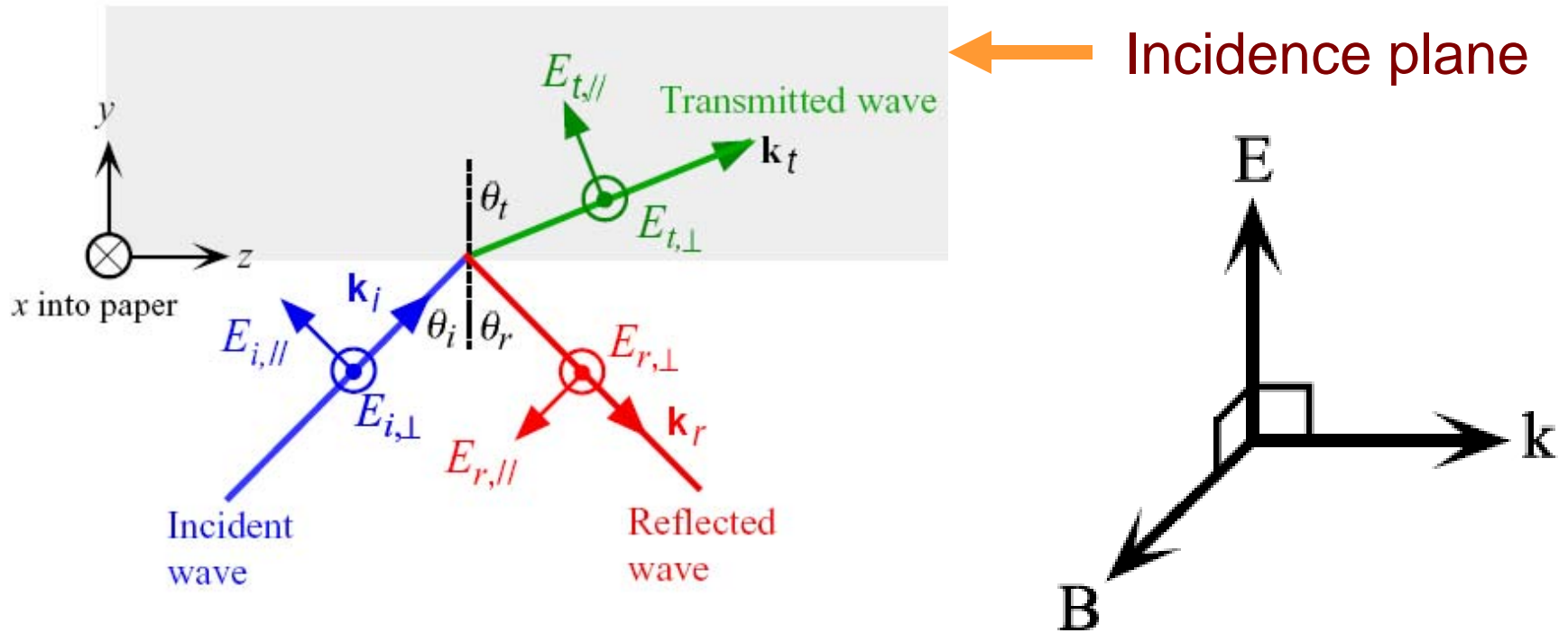
Our goal is to find E_{r0} and E_{t0} relative to E_{i0} , including any phase changes.



$$\left. \begin{aligned} \theta_i &= \theta_r \\ n_1 \sin \theta_i &= n_2 \sin \theta_t \end{aligned} \right\} \text{Snell's law}$$

$$\left. \begin{aligned} B_{\perp} &= (n/c)E_{\parallel} \\ B_{\parallel} &= (n/c)E_{\perp} \end{aligned} \right\} \text{Maxwell's equations}$$

$$\left. \begin{aligned} E_{\text{tangential}}(1) &= E_{\text{tangential}}(2) \\ B_{\text{tangential}}(1) &= B_{\text{tangential}}(2) \end{aligned} \right\} \text{Boundary conditions}$$



$$-E_{i,\perp} - E_{r,\perp} = -E_{t,\perp}$$

$$-B_{i,\perp} + B_{r,\perp} = -B_{t,\perp} \Rightarrow -(n_1/c)E_{i,\parallel} + (n_1/c)E_{r,\parallel} = -(n_2/c)E_{t,\parallel}$$

$$-E_{i,\parallel} \cos \theta_i - E_{r,\parallel} \cos \theta_r = -E_{t,\parallel} \cos \theta_t$$

$$B_{i,\parallel} \cos \theta_i - B_{r,\parallel} \cos \theta_r = B_{t,\parallel} \cos \theta_t \Rightarrow$$

$$(n_1/c)E_{i,\perp} \cos \theta_i - (n_1/c)E_{r,\perp} \cos \theta_r = (n_2/c)E_{t,\perp} \cos \theta_t$$

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases} \Rightarrow \begin{cases} x = \frac{de - bf}{ad - bc} \\ y = \frac{af - ce}{ad - bc} \end{cases}$$

$$\begin{cases} -E_{i,\perp} - E_{r,\perp} = -E_{t,\perp} \\ n_1 E_{i,\perp} \cos \theta_i - n_1 E_{r,\perp} \cos \theta_r = n_2 E_{t,\perp} \cos \theta_t \end{cases}$$

$$\Rightarrow \begin{cases} E_{r,\perp} - E_{t,\perp} = -E_{i,\perp} \\ n_1 E_{r,\perp} \cos \theta_r + n_2 E_{t,\perp} \cos \theta_t = n_1 E_{i,\perp} \cos \theta_i \end{cases}$$

$$\Rightarrow \begin{cases} E_{r,\perp} = \frac{-n_2 \cos \theta_t + n_1 \cos \theta_i}{n_2 \cos \theta_t + n_1 \cos \theta_r} E_{i,\perp} \\ E_{t,\perp} = \frac{n_1 \cos \theta_i + n_1 \cos \theta_r}{n_2 \cos \theta_t + n_1 \cos \theta_r} E_{i,\perp} \end{cases}$$

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases} \Rightarrow \begin{cases} x = \frac{de - bf}{ad - bc} \\ y = \frac{af - ce}{ad - bc} \end{cases}$$

$$\begin{cases} -n_1 E_{i,\parallel} + n_1 E_{r,\parallel} = -n_2 E_{t,\parallel} \\ -E_{i,\parallel} \cos \theta_i - E_{r,\parallel} \cos \theta_r = -E_{t,\parallel} \cos \theta_t \end{cases}$$

$$\Rightarrow \begin{cases} n_1 E_{r,\parallel} + n_2 E_{t,\parallel} = n_1 E_{i,\parallel} \\ E_{r,\parallel} \cos \theta_r - E_{t,\parallel} \cos \theta_t = -E_{i,\parallel} \cos \theta_i \end{cases}$$

$$\Rightarrow \begin{cases} E_{r,\parallel} = \frac{-n_1 \cos \theta_t + n_2 \cos \theta_i}{-n_1 \cos \theta_t - n_2 \cos \theta_r} E_{i,\parallel} \\ E_{t,\parallel} = \frac{-n_1 \cos \theta_i - n_1 \cos \theta_r}{-n_1 \cos \theta_t - n_2 \cos \theta_r} E_{i,\parallel} \end{cases}$$

$$\begin{cases} n = n_2 / n_1 \\ \theta_i = \theta_r \\ n_1 \sin \theta_i = n_2 \sin \theta_t \end{cases} \Rightarrow \sin \theta_i = n \sin \theta_t$$

$$\Rightarrow n \cos \theta_t = \left[n^2 - n^2 \sin^2 \theta_t \right]^{1/2} = \left[n^2 - \sin^2 \theta_i \right]^{1/2}$$

$$\begin{cases} E_{r,\perp} = \frac{-n_2 \cos \theta_t + n_1 \cos \theta_i}{n_2 \cos \theta_t + n_1 \cos \theta_r} E_{i,\perp} = \frac{\cos \theta_i - \left[n^2 - \sin^2 \theta_i \right]^{1/2}}{\cos \theta_i + \left[n^2 - \sin^2 \theta_i \right]^{1/2}} E_{i,\perp} \\ E_{t,\perp} = \frac{n_1 \cos \theta_i + n_1 \cos \theta_r}{n_2 \cos \theta_t + n_1 \cos \theta_r} E_{i,\perp} = \frac{2 \cos \theta_i}{\cos \theta_i + \left[n^2 - \sin^2 \theta_i \right]^{1/2}} E_{i,\perp} \end{cases}$$

$$\begin{cases} n = n_2 / n_1 \\ \theta_i = \theta_r \\ n_1 \sin \theta_i = n_2 \sin \theta_t \end{cases} \Rightarrow \sin \theta_i = n \sin \theta_t$$

$$\Rightarrow n \cos \theta_t = \left[n^2 - n^2 \sin^2 \theta_t \right]^{1/2} = \left[n^2 - \sin^2 \theta_i \right]^{1/2}$$

$$\begin{cases} E_{r,\parallel} = \frac{-n_1 \cos \theta_t + n_2 \cos \theta_i}{-n_1 \cos \theta_t - n_2 \cos \theta_r} E_{i,\parallel} = \frac{\left[n^2 - \sin^2 \theta_i \right]^{1/2} - n^2 \cos \theta_i}{\left[n^2 - \sin^2 \theta_i \right]^{1/2} + n^2 \cos \theta_i} E_{i,\parallel} \\ E_{t,\parallel} = \frac{-n_1 \cos \theta_i - n_1 \cos \theta_r}{-n_1 \cos \theta_t - n_2 \cos \theta_r} E_{i,\parallel} = \frac{2n \cos \theta_i}{\left[n^2 - \sin^2 \theta_i \right]^{1/2} + n^2 \cos \theta_i} E_{i,\parallel} \end{cases}$$

$$E_i = E_{i0} \exp j(\omega t - \mathbf{k}_i \bullet \mathbf{r})$$

$$E_r = E_{r0} \exp j(\omega t - \mathbf{k}_r \bullet \mathbf{r})$$

$$E_t = E_{t0} \exp j(\omega t - \mathbf{k}_t \bullet \mathbf{r})$$

r_{\perp} and r_{\parallel} are reflection coefficients. t_{\perp} and t_{\parallel} are transmission coefficients.

$$r_{\perp} = \frac{E_{r0,\perp}}{E_{i0,\perp}} = \frac{\cos \theta_i - [n^2 - \sin^2 \theta_i]^{1/2}}{\cos \theta_i + [n^2 - \sin^2 \theta_i]^{1/2}}$$

$$t_{\perp} = \frac{E_{t0,\perp}}{E_{i0,\perp}} = \frac{2 \cos \theta_i}{\cos \theta_i + [n^2 - \sin^2 \theta_i]^{1/2}}$$

$$r_{\parallel} = \frac{E_{r0,\parallel}}{E_{i0,\parallel}} = \frac{[n^2 - \sin^2 \theta_i]^{1/2} - n^2 \cos \theta_i}{[n^2 - \sin^2 \theta_i]^{1/2} + n^2 \cos \theta_i}$$

$$t_{\parallel} = \frac{E_{t0,\parallel}}{E_{i0,\parallel}} = \frac{2n \cos \theta_i}{[n^2 - \sin^2 \theta_i]^{1/2} + n^2 \cos \theta_i}$$

$$r_{\perp} = |r_{\perp}| \exp(-j\phi_{r,\perp})$$

$$r_{\parallel} = |r_{\parallel}| \exp(-j\phi_{r,\parallel})$$

$$t_{\perp} = |t_{\perp}| \exp(-j\phi_{t,\perp})$$

$$t_{\parallel} = |t_{\parallel}| \exp(-j\phi_{t,\parallel})$$

Complex coefficients can only be obtained when $n^2 - \sin^2 \theta_i < 0$.



$n_1 > n_2, \theta_i > \theta_c$
Phase changes other than 0 or 180° occur only when there is TIR.

$$r_{\parallel} + nt_{\parallel} = 1$$

$$r_{\perp} + 1 = t_{\perp}$$

$$r_{\perp} = \frac{\cos \theta_i - [n^2 - \sin^2 \theta_i]^{1/2}}{\cos \theta_i + [n^2 - \sin^2 \theta_i]^{1/2}}$$

$$r_{\parallel} = \frac{[n^2 - \sin^2 \theta_i]^{1/2} - n^2 \cos \theta_i}{[n^2 - \sin^2 \theta_i]^{1/2} + n^2 \cos \theta_i}$$

When $\theta_i = 0$ (normal incidence):

$$r_{\perp} = \frac{1-n}{1+n} = \frac{n_1 - n_2}{n_1 + n_2}$$

$$r_{\parallel} = \frac{n - n^2}{n + n^2} = \frac{n_1 - n_2}{n_1 + n_2}$$

When $r_{\parallel} = 0$ (reflected light is polarized):

$$[n^2 - \sin^2 \theta_p]^{1/2} = n^2 \cos \theta_p \Rightarrow n^2 - \sin^2 \theta_p = n^4 \cos^2 \theta_p$$

$$\Rightarrow \frac{n^2}{\cos^2 \theta_p} - \frac{\sin^2 \theta_p}{\cos^2 \theta_p} = n^4 \Rightarrow n^2(1 + \tan^2 \theta_p) - \tan^2 \theta_p = n^4$$

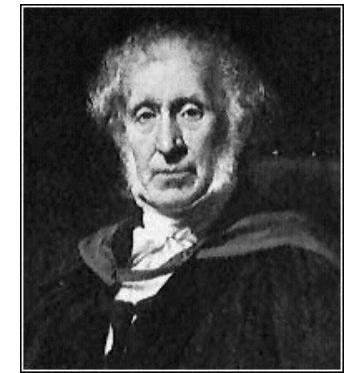
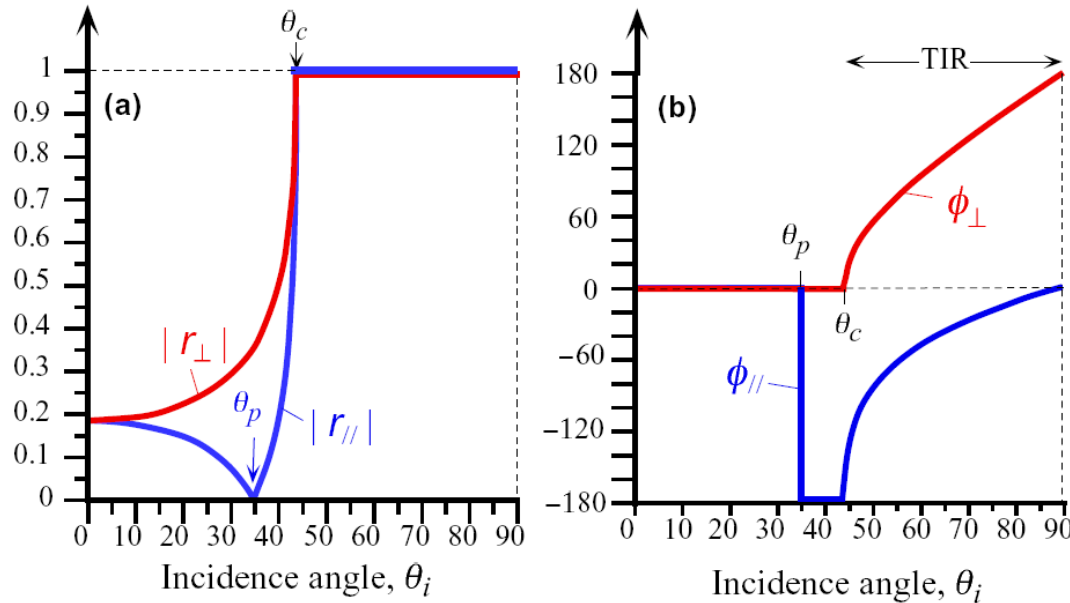
$$\Rightarrow \tan \theta_p = \frac{n_2}{n_1}$$

θ_p is called the polarization angle or Brewster's angle.

$$\sin \theta_c = \frac{n_2}{n_1}$$

Brewster's Angle (Polarization Angle)

$$n_1 = 1.44, n_2 = 1.00$$



David Brewster
(1781-1868)

$$\theta_p = 34.78^\circ$$

$$\theta_c = 43.98^\circ$$

- ❖ When $\theta_i = \theta_p$, reflected light is linearly polarized and is perpendicular to the incidence plane.
- ❖ When $\theta_p < \theta_i < \theta_c$, there is a phase shift of 180° in r_{\parallel} .
- ❖ $|r_{\perp}|$ and $|r_{\parallel}|$ increases with θ_i when $\theta_i > \theta_p$.
- ❖ There are phase shifts (other than 0 or 180°) in r_{\perp} and r_{\parallel} when $\theta_i > \theta_c$.

Example: calculate r_{\perp} and r_{\parallel} when $n_1 = 1.44$, $n_2 = 1.00$, and $\theta_i = 40^\circ$.

Solution:

$$n = \frac{n_2}{n_1} = \frac{1.00}{1.44} = 0.694$$

$$r_{\perp} = \frac{\cos \theta_i - [n^2 - \sin^2 \theta_i]^{1/2}}{\cos \theta_i + [n^2 - \sin^2 \theta_i]^{1/2}}$$
$$= \frac{\cos 40^\circ - [0.694^2 - \sin^2 40^\circ]^{1/2}}{\cos 40^\circ + [0.694^2 - \sin^2 40^\circ]^{1/2}} = 0.490$$

$$r_{\parallel} = \frac{[n^2 - \sin^2 \theta_i]^{1/2} - n^2 \cos \theta_i}{[n^2 - \sin^2 \theta_i]^{1/2} + n^2 \cos \theta_i}$$
$$= \frac{[0.694^2 - \sin^2 40^\circ]^{1/2} - 0.694^2 \cos 40^\circ}{[0.694^2 - \sin^2 40^\circ]^{1/2} + 0.694^2 \cos 40^\circ} = -0.170$$

Example: calculate r_{\perp} and r_{\parallel} when $n_1 = 1.00$, $n_2 = 1.44$, and $\theta_i = 40^\circ$.

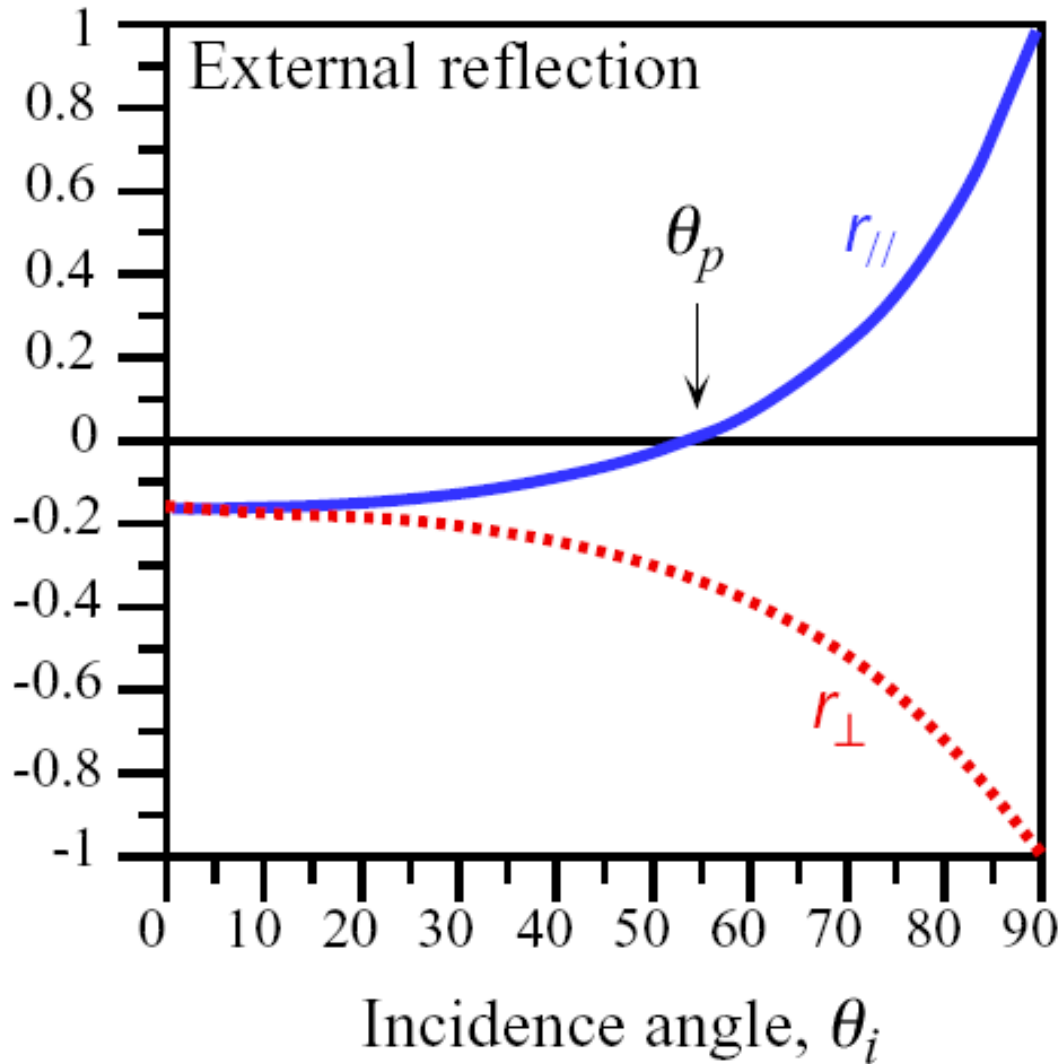
Solution:

$$n = \frac{n_2}{n_1} = \frac{1.44}{1.00} = 1.44$$

$$\begin{aligned} r_{\perp} &= \frac{\cos \theta_i - [n^2 - \sin^2 \theta_i]^{1/2}}{\cos \theta_i + [n^2 - \sin^2 \theta_i]^{1/2}} \\ &= \frac{\cos 40^\circ - [1.44^2 - \sin^2 40^\circ]^{1/2}}{\cos 40^\circ + [1.44^2 - \sin^2 40^\circ]^{1/2}} = -0.255 \end{aligned}$$

$$\begin{aligned} r_{\parallel} &= \frac{[n^2 - \sin^2 \theta_i]^{1/2} - n^2 \cos \theta_i}{[n^2 - \sin^2 \theta_i]^{1/2} + n^2 \cos \theta_i} \\ &= \frac{[1.44^2 - \sin^2 40^\circ]^{1/2} - 1.44^2 \cos 40^\circ}{[1.44^2 - \sin^2 40^\circ]^{1/2} + 1.44^2 \cos 40^\circ} = -0.104 \end{aligned}$$

$$n_1 = 1.00, n_2 = 1.44$$



$$\tan \theta_p = \frac{n_2}{n_1}$$

$$\theta_p = 55.22^\circ$$

What are the phase changes at TIR ($n_1 > n_2$ and $\theta_i > \theta_c$)?

$$r_{\perp} = \frac{\cos \theta_i - [n^2 - \sin^2 \theta_i]^{1/2}}{\cos \theta_i + [n^2 - \sin^2 \theta_i]^{1/2}} \quad |r_{\perp}| = 1$$

$$r_{//} = \frac{[n^2 - \sin^2 \theta_i]^{1/2} - n^2 \cos \theta_i}{[n^2 - \sin^2 \theta_i]^{1/2} + n^2 \cos \theta_i} \quad |r_{//}| = 1$$

$$z = x - jy$$

$$z = |z| \exp(-j\phi)$$

$$|z| = (x^2 + y^2)^{1/2}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$r_{\perp} = \frac{\cos \theta_i - j[\sin^2 \theta_i - n^2]^{1/2}}{\cos \theta_i + j[\sin^2 \theta_i - n^2]^{1/2}}$$

$$r_{//} = \frac{-n^2 \cos \theta_i + j[\sin^2 \theta_i - n^2]^{1/2}}{n^2 \cos \theta_i + j[\sin^2 \theta_i - n^2]^{1/2}}$$

$$z = \frac{z_1}{z_2} = \frac{|z_1| \exp(-j\phi_1)}{|z_2| \exp(-j\phi_2)}$$

$$= \frac{|z_1|}{|z_2|} \exp[-j(\phi_1 - \phi_2)]$$

$$r_{\perp} = \frac{\cos \theta_i - j[\sin^2 \theta_i - n^2]^{1/2}}{\cos \theta_i + j[\sin^2 \theta_i - n^2]^{1/2}}$$

$$\phi_1 = \tan^{-1} \left(\frac{[\sin^2 \theta_i - n^2]^{1/2}}{\cos \theta_i} \right)$$

$$\phi_2 = -\tan^{-1} \left(\frac{[\sin^2 \theta_i - n^2]^{1/2}}{\cos \theta_i} \right)$$

$$\phi_{\perp} = \phi_1 - \phi_2 = 2 \tan^{-1} \left(\frac{[\sin^2 \theta_i - n^2]^{1/2}}{\cos \theta_i} \right)$$

$$\tan \left(\frac{1}{2} \phi_{\perp} \right) = \frac{[\sin^2 \theta_i - n^2]^{1/2}}{\cos \theta_i}$$

$$r_{//} = \frac{-n^2 \cos \theta_i + j[\sin^2 \theta_i - n^2]^{1/2}}{n^2 \cos \theta_i + j[\sin^2 \theta_i - n^2]^{1/2}} \quad \phi_1 = \tan^{-1} \left(\frac{[\sin^2 \theta_i - n^2]^{1/2}}{n^2 \cos \theta_i} \right) - \pi$$

$$\phi_2 = -\tan^{-1} \left(\frac{[\sin^2 \theta_i - n^2]^{1/2}}{n^2 \cos \theta_i} \right)$$

$$\phi_{//} = \phi_1 - \phi_2 = 2 \tan^{-1} \left(\frac{[\sin^2 \theta_i - n^2]^{1/2}}{n^2 \cos \theta_i} \right) - \pi$$

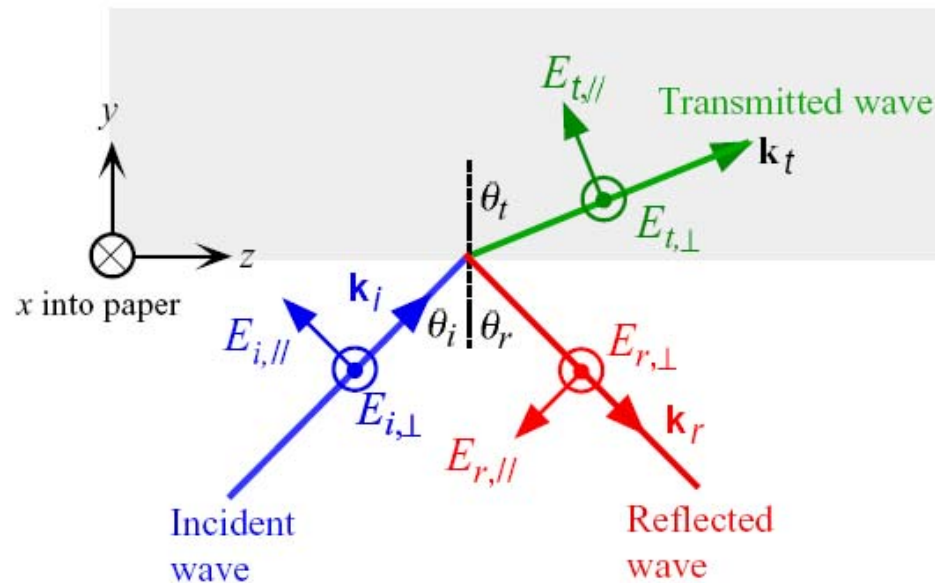
$$\phi_{//} + \pi = 2 \tan^{-1} \left(\frac{[\sin^2 \theta_i - n^2]^{1/2}}{n^2 \cos \theta_i} \right)$$

$$\tan \left(\frac{1}{2} \phi_{//} + \frac{1}{2} \pi \right) = \frac{[\sin^2 \theta_i - n^2]^{1/2}}{n^2 \cos \theta_i}$$

Consider now:

$$E_t = E_{t0} \exp j(\omega t - \mathbf{k}_t \cdot \mathbf{r})$$

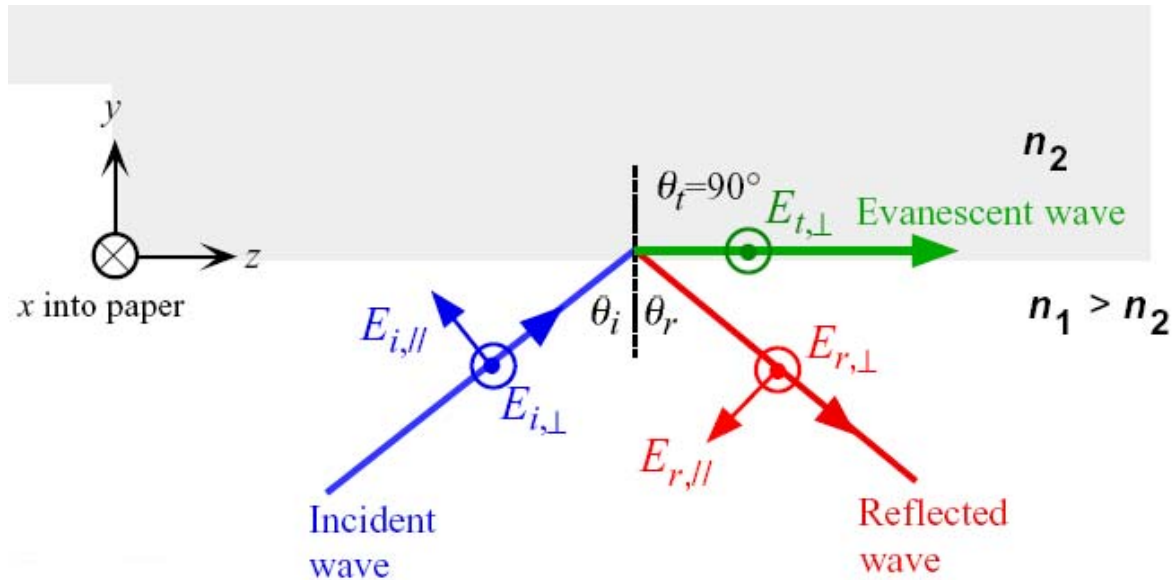
$$= E_{t0} \exp j(\omega t - k_{ty} y - k_{tz} z)$$



$$k_{tz} = k_t \sin \theta_t = \frac{2\pi}{\lambda} n_2 \sin \theta_t = \frac{2\pi}{\lambda} n_1 \sin \theta_i = k_i \sin \theta_i = k_{iz}$$

$$k_{ty} = k_t \cos \theta_t = \frac{2\pi}{\lambda} n_2 [1 - \sin^2 \theta_t]^{1/2} = \frac{2\pi}{\lambda} n_2 \left[1 - \left(\frac{n_1}{n_2} \right)^2 \sin^2 \theta_i \right]^{1/2}$$

Evanescent Wave



At TIR: $\theta_i > \theta_c \Rightarrow \left(\frac{n_1}{n_2}\right)^2 \sin^2 \theta_i > \left(\frac{n_1}{n_2}\right)^2 \sin^2 \theta_c = 1$

$$k_{ty} = -j \frac{2\pi}{\lambda} n_2 \left[\left(\frac{n_1}{n_2}\right)^2 \sin^2 \theta_i - 1 \right]^{1/2} = -j\alpha_2$$

Why do we put a minus here?

Evanescent Wave

$$\alpha_2 = \frac{2\pi n_2}{\lambda} \left[\left(\frac{n_1}{n_2} \right)^2 \sin^2 \theta_i - 1 \right]^{1/2}$$

$$\begin{aligned} E_t(y, z, t) &= E_{t0} \exp j(\omega t - k_{ty} y - k_{tz} z) \\ &= E_{t0} \exp j(-k_{ty} y) \exp j(\omega t - k_{iz} z) \\ &= E_{t0} \exp(-\alpha_2 y) \exp j(\omega t - k_{iz} z) \end{aligned}$$

$$\delta = \frac{1}{\alpha_2} = \frac{\lambda}{2\pi} \left[n_1^2 \sin^2 \theta_i - n_2^2 \right]^{-1/2}$$

the penetration depth

$\lambda = 514.5$ nm (green light)

$n_1 = 1.44$ (glass)

$n_2 = 1.00$ (air)

$\theta_i = 50^\circ$ ($> \theta_c = 43.98^\circ$)

$$\delta = \frac{514.5}{2\pi} \left[1.44^2 \sin^2(50^\circ) - 1.00^2 \right]^{-1/2} = 176 \text{ nm}$$

Total Internal Reflection Fluorescence (TIRF) Microscopy

Total Internal Reflection Fluorescence

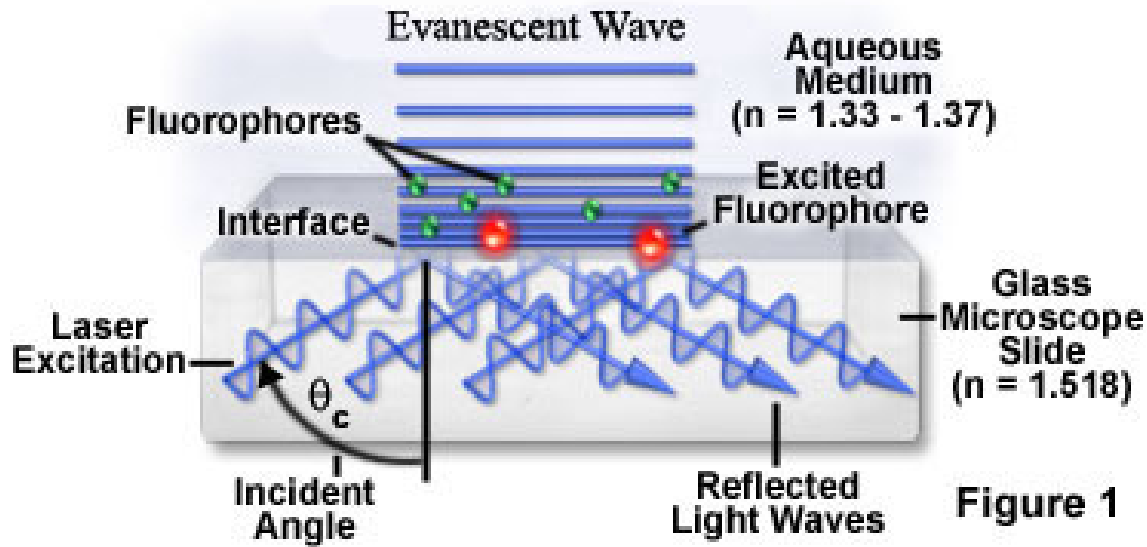
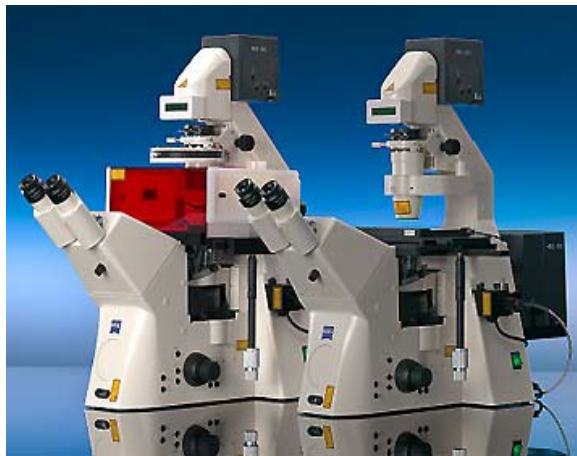
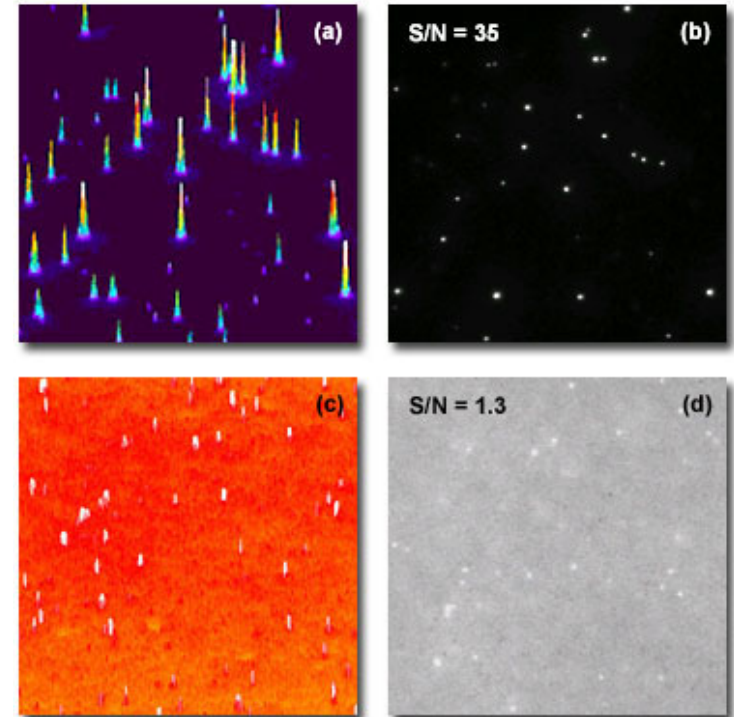


Figure 1

Microspheres in TIR and Widefield Fluorescence



Carl Zeiss

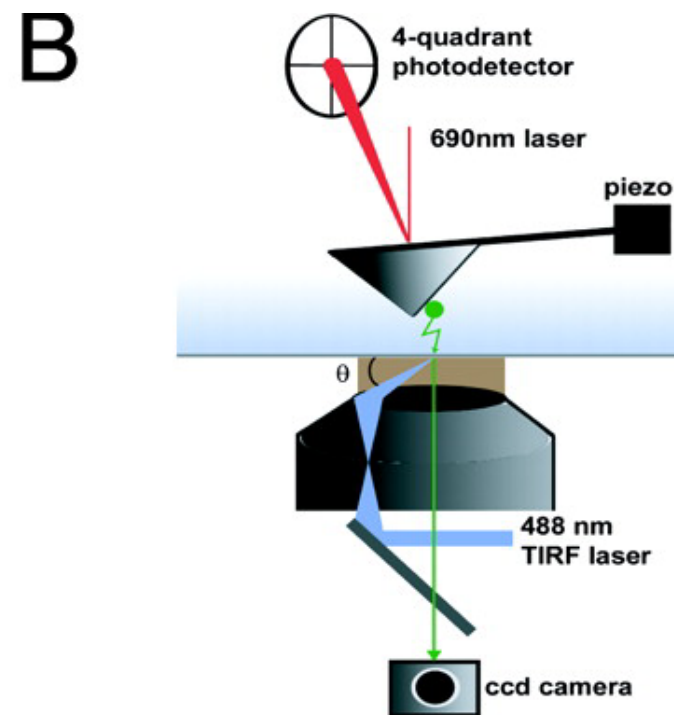
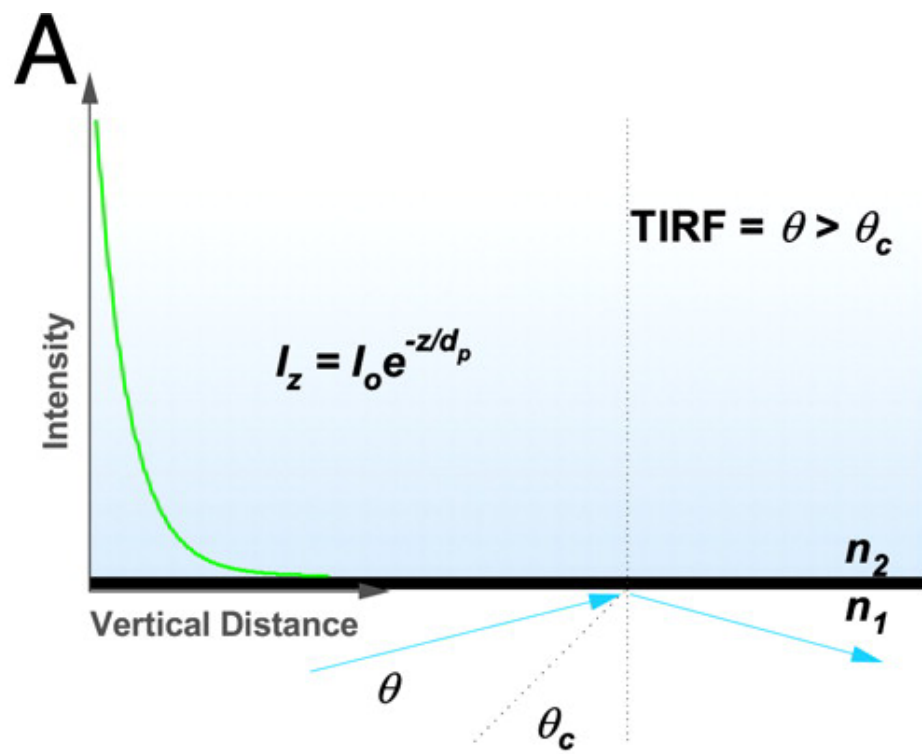
TIR illumination can greatly improve signal-to-noise ratios in applications requiring imaging of minute structures or single molecules in specimens having large numbers of fluorophores located outside of the optical plane of interest, such as molecules in solution in Brownian motion, vesicles undergoing endocytosis or exocytosis, or single protein trafficking in cells.

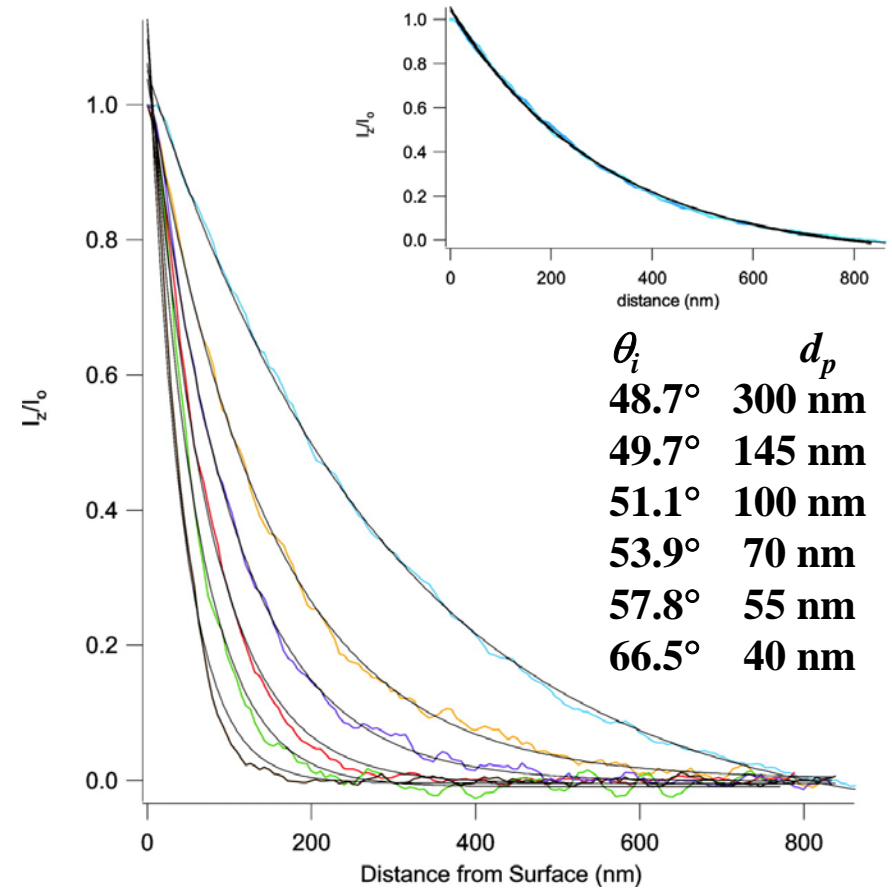
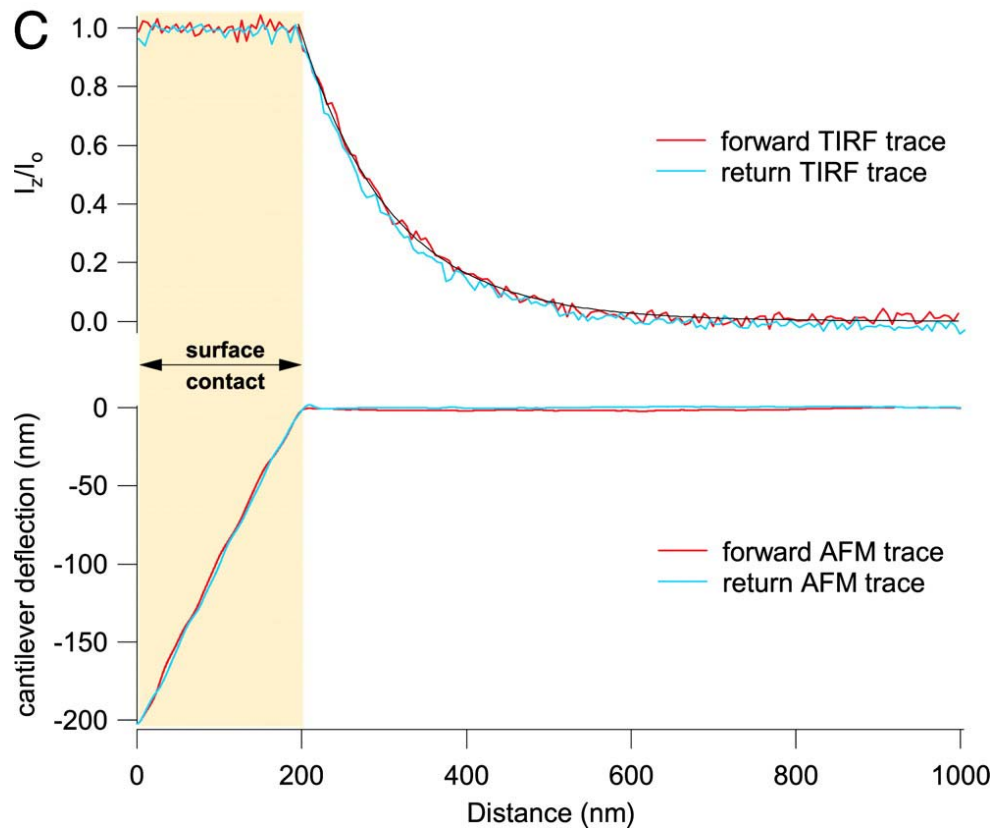
Simultaneous atomic force microscope and fluorescence measurements of protein unfolding using a calibrated evanescent wave

Atom Sarkar^{**†}, Ragan B. Robertson^{*}, and Julio M. Fernandez^{**‡}

^{*}Department of Biological Sciences, Columbia University, New York, NY 10025; and [†]Department of Neurological Surgery, Mayo Clinic College of Medicine, Rochester, MN 55905

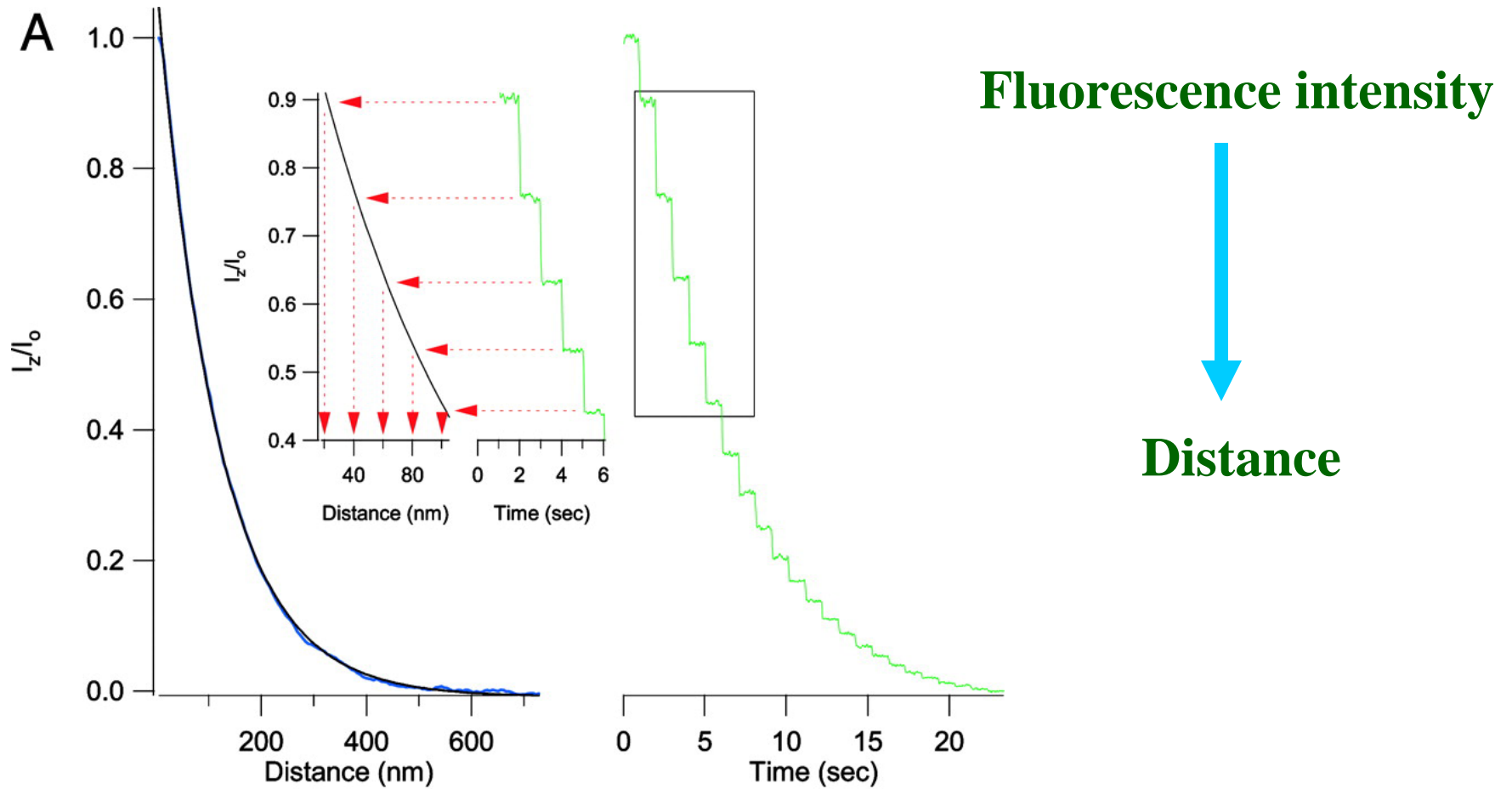
12882–12886 | PNAS | August 31, 2004 | vol. 101 | no. 35





$$I_z = I_0 e^{-z/d_p}$$

$$d_p = \frac{\lambda}{4\pi} \left[n_1^2 \sin^2 \theta_i - n_2^2 \right]^{-1/2}$$



Evanescent nanometry

Achieves a resolution of ~ 1 nm.

Requires highly photostable fluorescent materials.

Intensity, Reflectance, and Transmittance

Light intensity

$$I = \frac{1}{2} v \epsilon_r \epsilon_o E_0^2 \propto n E_0^2$$

Reflectance

$$R_{\perp} = \frac{|E_{r0,\perp}|^2}{|E_{i0,\perp}|^2} = |r_{\perp}|^2 \quad R_{//} = \frac{|E_{r0,//}|^2}{|E_{i0,//}|^2} = |r_{//}|^2$$

Transmittance

$$T_{\perp} = \frac{n_2 |E_{t0,\perp}|^2}{n_1 |E_{i0,\perp}|^2} = \left(\frac{n_2}{n_1} \right) |t_{\perp}|^2 \quad T_{//} = \frac{n_2 |E_{t0,//}|^2}{n_1 |E_{i0,//}|^2} = \left(\frac{n_2}{n_1} \right) |t_{//}|^2$$

When $\theta_i = 0$ (normal incidence):

$$r_{\perp} = \frac{E_{r0,\perp}}{E_{i0,\perp}} = \frac{\cos \theta_i - [n^2 - \sin^2 \theta_i]^{1/2}}{\cos \theta_i + [n^2 - \sin^2 \theta_i]^{1/2}}$$

$$r_{//} = \frac{E_{r0,//}}{E_{i0,//}} = \frac{[n^2 - \sin^2 \theta_i]^{1/2} - n^2 \cos \theta_i}{[n^2 - \sin^2 \theta_i]^{1/2} + n^2 \cos \theta_i}$$

$$t_{\perp} = \frac{E_{t0,\perp}}{E_{i0,\perp}} = \frac{2 \cos \theta_i}{\cos \theta_i + [n^2 - \sin^2 \theta_i]^{1/2}}$$

$$t_{//} = \frac{E_{t0,//}}{E_{i0,//}} = \frac{2n \cos \theta_i}{[n^2 - \sin^2 \theta_i]^{1/2} + n^2 \cos \theta_i}$$

$$= \frac{1-n}{1+n} = \frac{n_1-n_2}{n_1+n_2}$$

$$= \frac{n-n^2}{n+n^2} = \frac{n_1-n_2}{n_1+n_2}$$

$$= \frac{2}{1+n} = \frac{2n_1}{n_1+n_2}$$

$$= \frac{2n}{n+n^2} = \frac{2n_1}{n_1+n_2}$$

$$R = R_{\perp} = R_{//} = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2$$

$$T = T_{\perp} = T_{//} = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

Example: A ray of light traveling in a glass medium of refractive index $n_1 = 1.450$ becomes incident on a less dense glass medium of refractive index $n_2 = 1.430$. Suppose the free space wavelength (λ) of the light ray is $1 \mu\text{m}$. (a) What is the critical angle for TIR? (b) What is the phase change in the reflected wave when $\theta_i = 85^\circ$ and 90° ? (c) What is the penetration depth of the evanescent wave into medium 2 when $\theta_i = 85^\circ$ and 90° ?

Solution:

$$\sin \theta_c = n_2 / n_1 = 1.430 / 1.450 \Rightarrow \theta_c = 80.47^\circ$$

$$\tan\left(\frac{1}{2}\phi_\perp\right) = \frac{[\sin^2 \theta_i - n^2]^{1/2}}{\cos \theta_i} = \frac{\left[\sin^2 85^\circ - \left(\frac{1.430}{1.450}\right)^2\right]^{1/2}}{\cos 85^\circ} = 1.614 \Rightarrow \phi_\perp = 116.45^\circ \quad \theta_i = 90^\circ$$

$$\tan\left(\frac{1}{2}\phi_\parallel + \frac{1}{2}\pi\right) = \frac{[\sin^2 \theta_i - n^2]^{1/2}}{n^2 \cos \theta_i} = \frac{\left[\sin^2 85^\circ - \left(\frac{1.430}{1.450}\right)^2\right]^{1/2}}{\left(\frac{1.430}{1.450}\right)^2 \cos 85^\circ} = 1.660 \Rightarrow \phi_\parallel = -62.13^\circ \quad \phi_\perp = 180^\circ$$

$$\phi_\parallel = 0^\circ$$

$$\delta = \frac{\lambda}{2\pi} [n_1^2 \sin^2 \theta_i - n_2^2]^{1/2} = \frac{1\mu\text{m}}{2\pi} [1.450^2 \sin^2 85^\circ - 1.430^2]^{1/2} = 0.780\mu\text{m}$$

Example: Consider a light ray at normal incidence on a boundary between a glass medium of refractive index 1.5 and air of refractive index 1.0. (a) If light is traveling from air to glass, what are the reflectance and transmittance? (b) If light is traveling from glass to air, what are the reflectance and transmittance? (c) What is the Brewster's angle when light is traveling from air to glass?

Solution:

$$\begin{cases} n_1 = 1.0 \\ n_2 = 1.5 \end{cases} \Rightarrow \begin{cases} R = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2 = \left(\frac{1.0 - 1.5}{1.0 + 1.5} \right)^2 = 0.04 \\ T = \frac{4n_1n_2}{(n_1 + n_2)^2} = \frac{4 \times 1.0 \times 1.5}{(1.0 + 1.5)^2} = 0.96 \end{cases}$$

$$\begin{cases} n_1 = 1.5 \\ n_2 = 1.0 \end{cases} \Rightarrow \begin{cases} R = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2 = \left(\frac{1.5 - 1.0}{1.5 + 1.0} \right)^2 = 0.04 \\ T = \frac{4n_1n_2}{(n_1 + n_2)^2} = \frac{4 \times 1.5 \times 1.0}{(1.5 + 1.0)^2} = 0.96 \end{cases}$$

$$\tan \theta_p = \frac{n_2}{n_1} = \frac{1.5}{1.0} \Rightarrow \theta_p = 56.31^\circ$$

Light will lose 4% of its intensity whenever it travels through an interface between air and glass.

Antireflection Coatings

Current commercial solar cells are made of silicon ($n \approx 3.5$ at wavelengths of 700–800 nm).

$$R = \frac{(n_1 - n_2)^2}{(n_1 + n_2)^2} = \frac{(1.0 - 3.5)^2}{(1.0 + 3.5)^2} = 0.31$$

$$\Delta_c \phi = k_c 2d = \frac{2\pi n_2}{\lambda} (2d) = m\pi$$

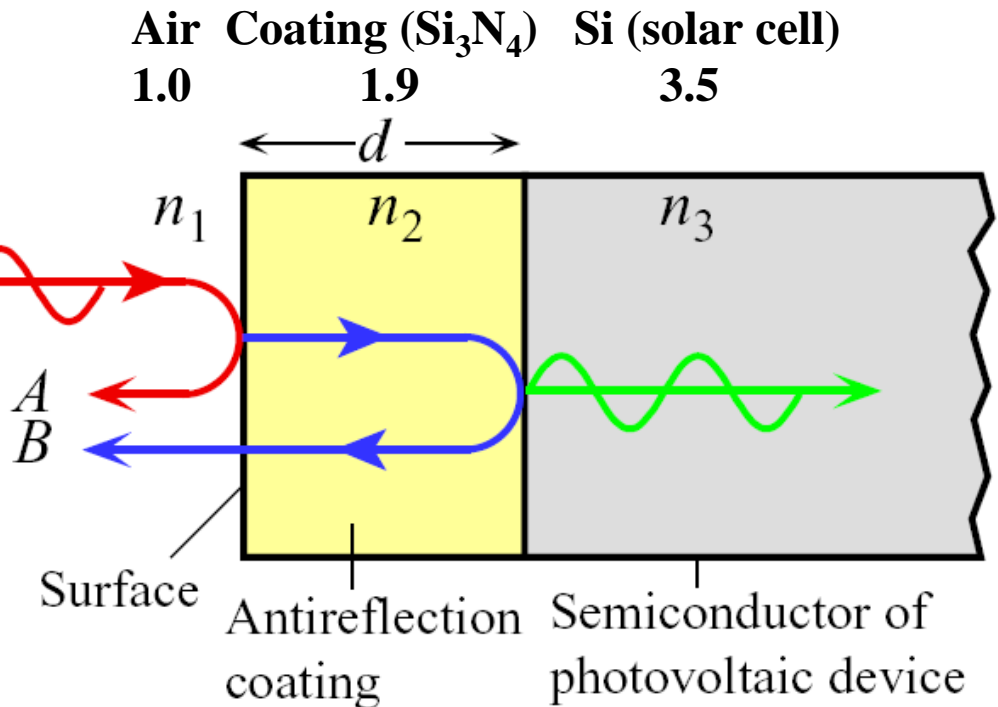
$$\Rightarrow d = m \left(\frac{\lambda}{4n_2} \right)$$

$r_{12} = r_{23}$ in order to obtain efficient destructive interference.

$$\frac{n_1 - n_2}{n_1 + n_2} = \frac{n_2 - n_3}{n_2 + n_3} \Rightarrow n_2 = \sqrt{n_1 n_3}$$

$$n_2 = \sqrt{n_1 n_3} = \sqrt{1.0 \times 3.5} = 1.87$$

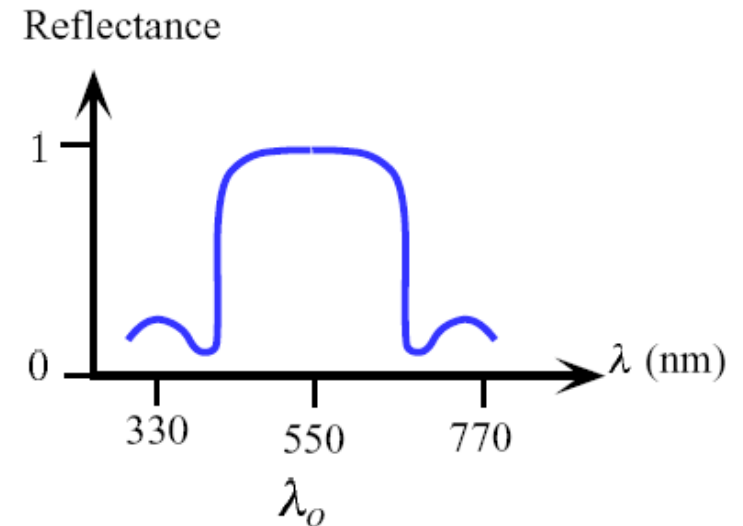
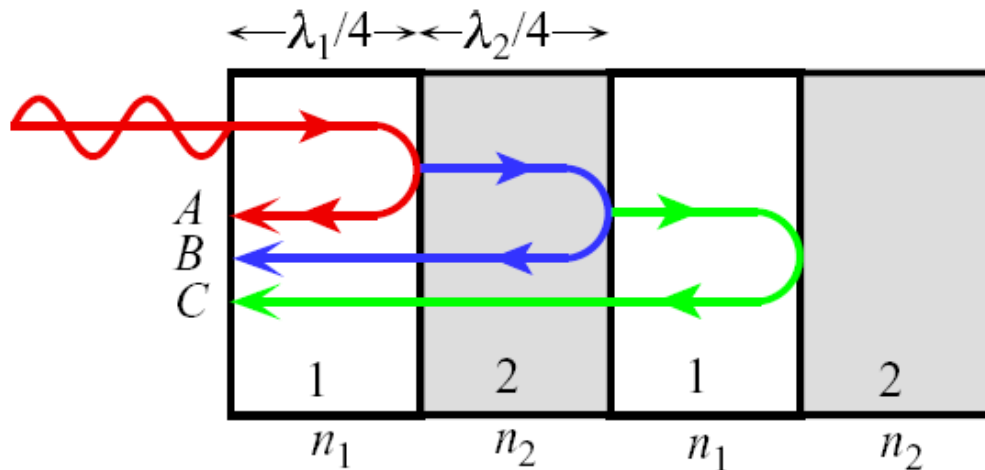
$$d = \frac{\lambda}{4n_2} = \frac{700\text{nm}}{4 \times 1.9} = 92\text{nm}$$



$$r_{12,\perp} = r_{12,\parallel} = -0.31$$

$$r_{23,\perp} = r_{23,\parallel} = -0.30$$

Dielectric Mirrors



$$n_1 < n_2$$

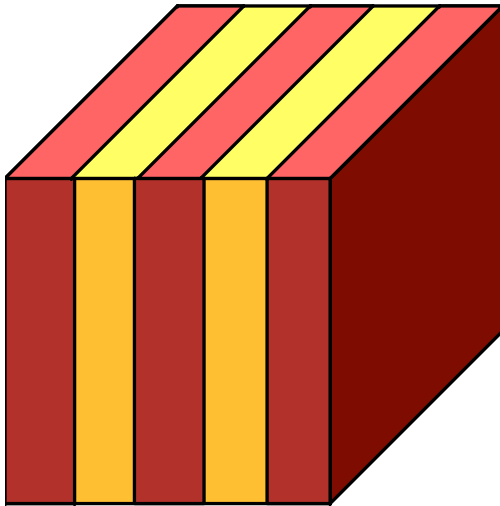
$$r_{12} = \frac{n_1 - n_2}{n_1 + n_2} < 0 \quad t_{12} = \frac{2n_1}{n_1 + n_2} > 0$$

$$r_{21} = \frac{n_2 - n_1}{n_2 + n_1} > 0 \quad t_{21} = \frac{2n_2}{n_2 + n_1} > 0$$

We can find that A, B, and C waves are in phase with each other. They interfere with each other constructively. The total reflectance will be close to unity.

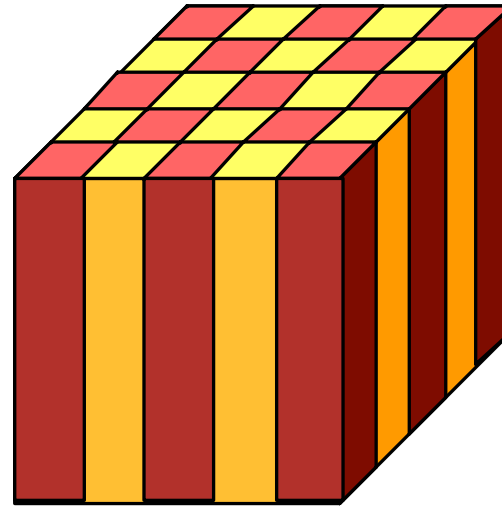
Photonic Crystals

1-D



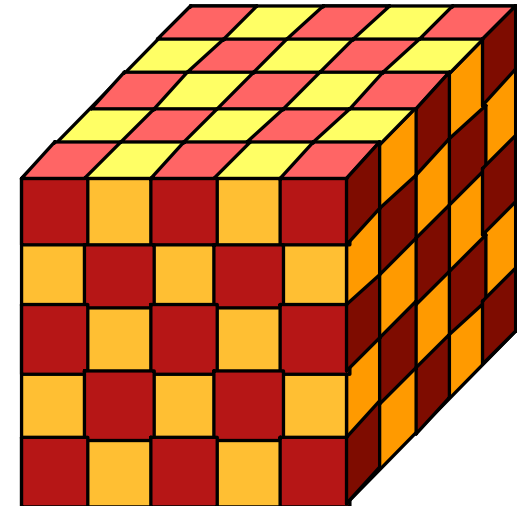
periodic in
one direction

2-D



periodic in
two directions

3-D

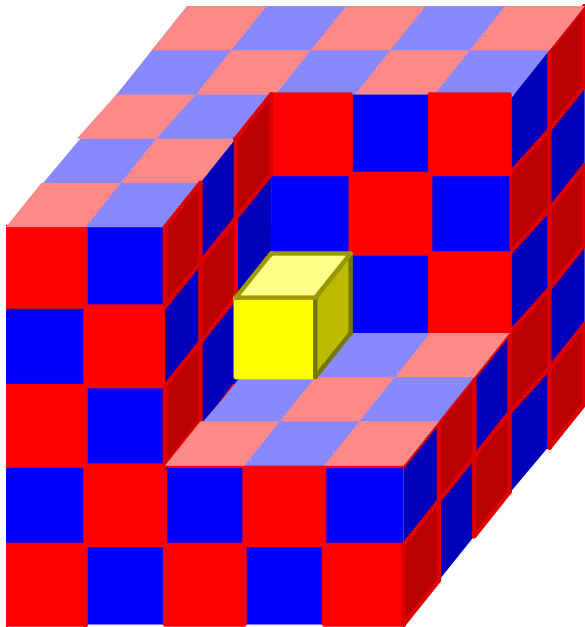


periodic in
three directions

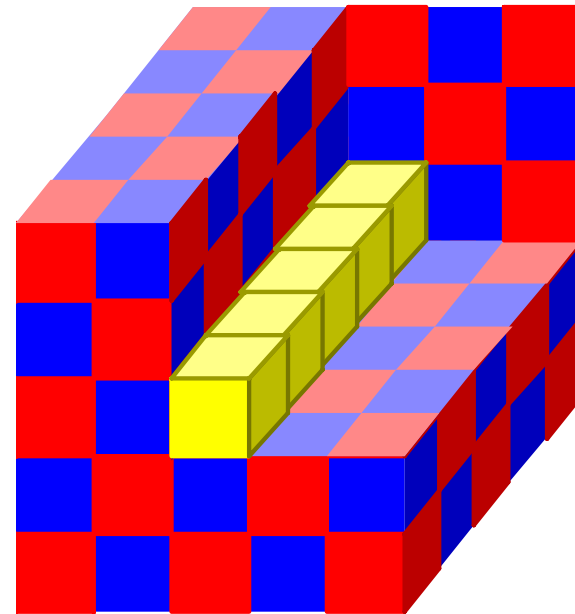
Photonic crystals are ordered structures in which two media with different dielectric constants or refractive indices are arranged in a periodic form, with lattice constants comparable to the wavelength of light. The light propagation is completely forbidden within photonic crystals if they are properly designed (refractive index contrast, crystal structure, and lattice constant).

Defects in Photonic Crystals

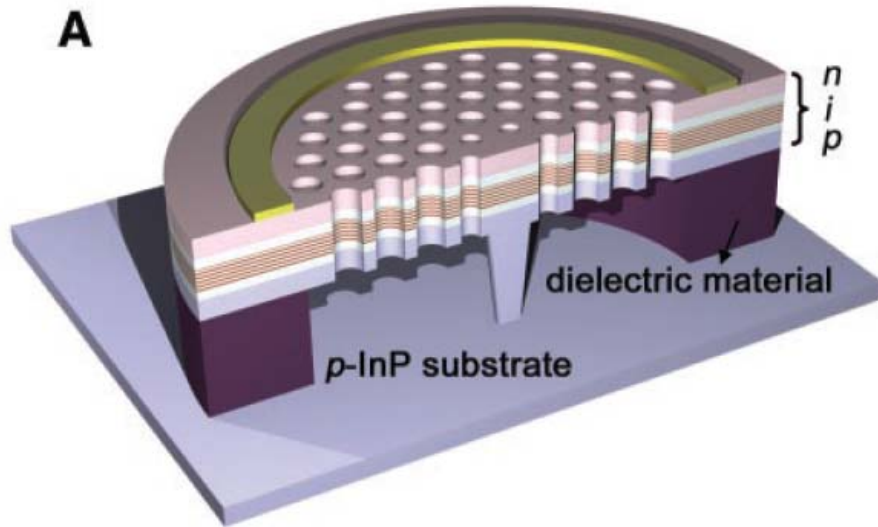
cavity



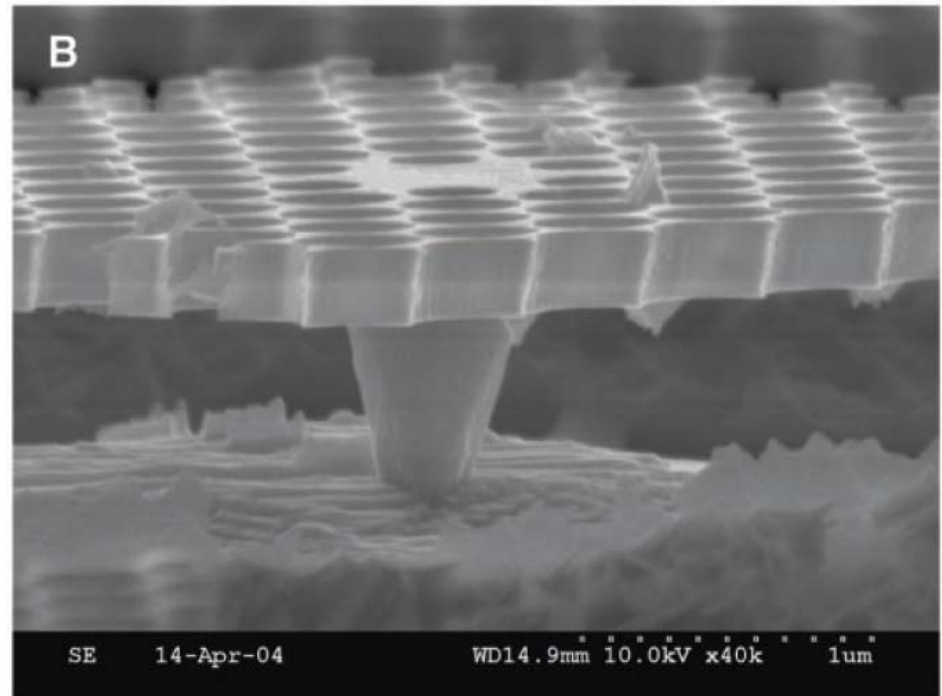
waveguide



Low-Pumping-Current Lasers

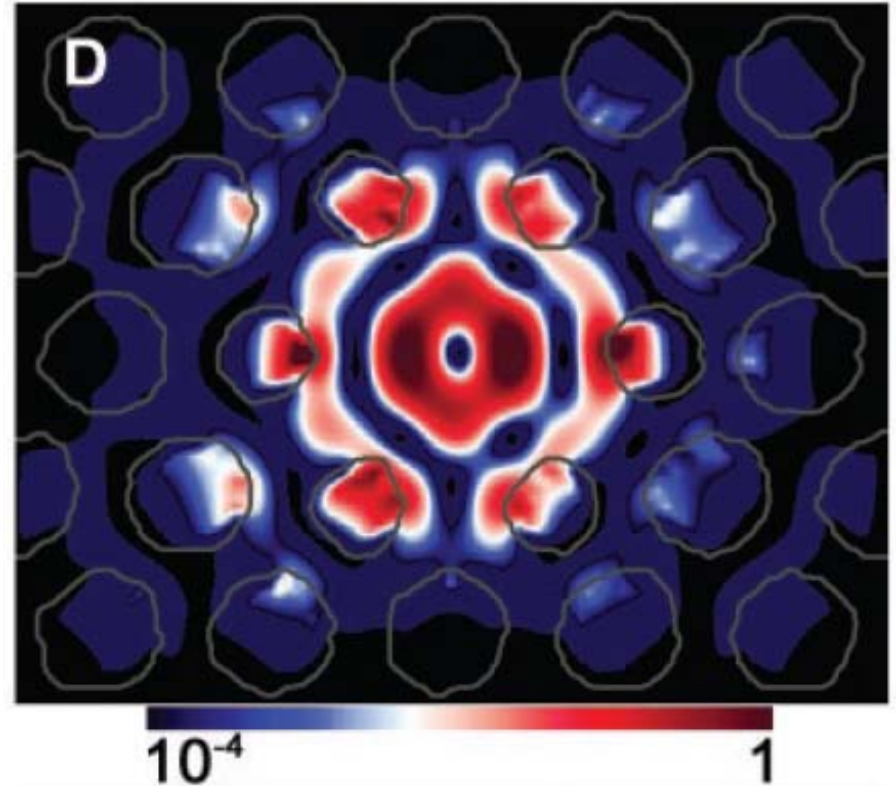
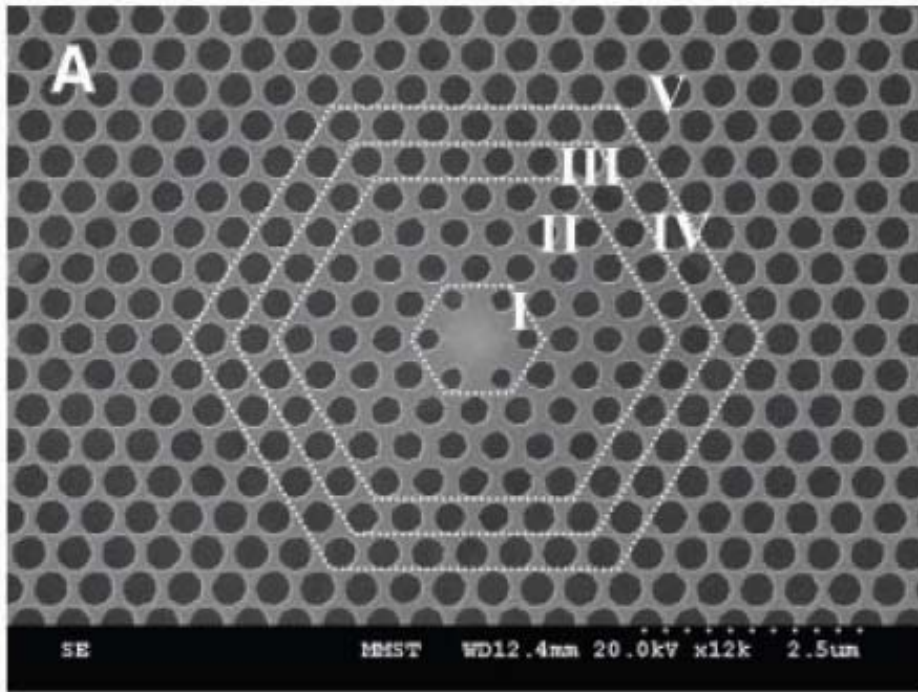


SEM image



H.-G. Park, S.-H. Kim, S.-H. Kwon, Y.-G. Ju,
J.-K. Yang, J.-H. Baek, S.-B. Kim, Y.-H. Lee,
Science **2004**, 305, 1444.

Low-Pumping-Current Lasers



radii of air holes:

$0.28a$ (I)

$0.35a$ (II)

$0.385a$ (III)

$0.4a$ (IV)

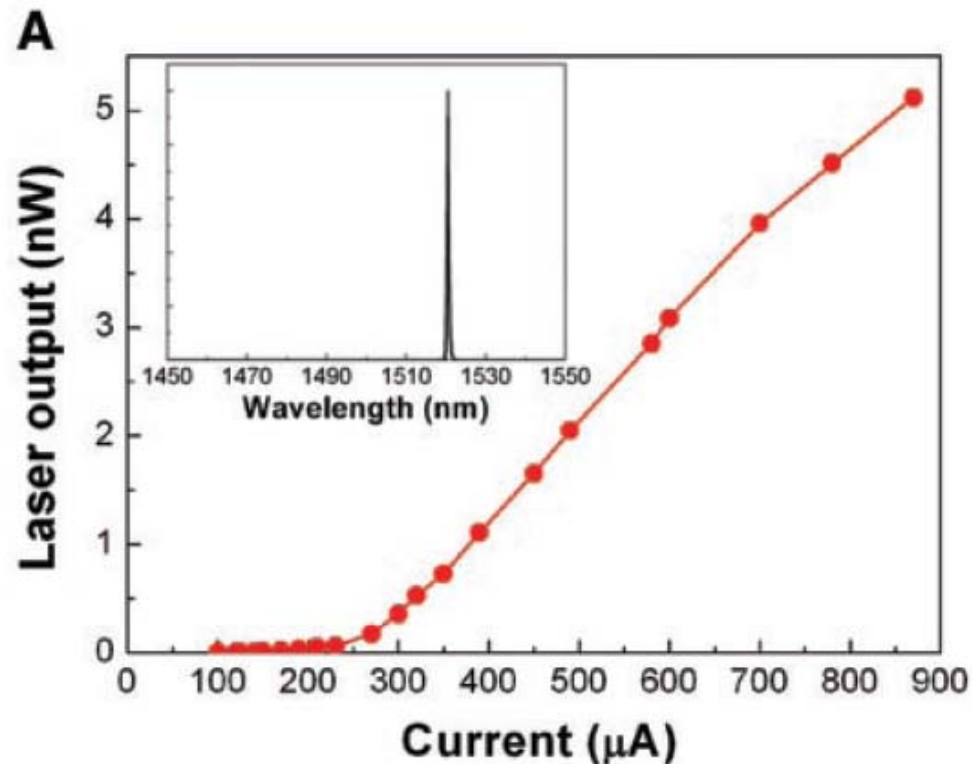
$0.41a$ (V)



Designed according to FDTD (finite difference time domain) calculations.

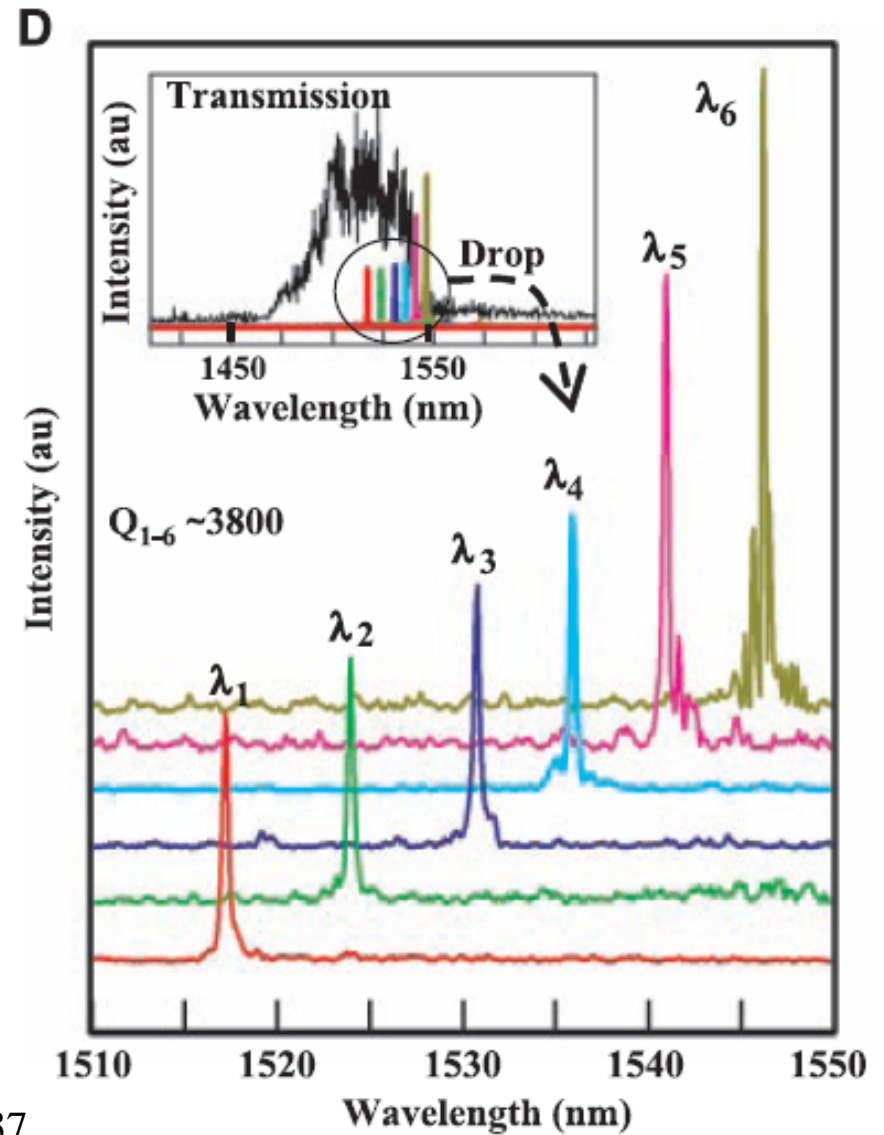
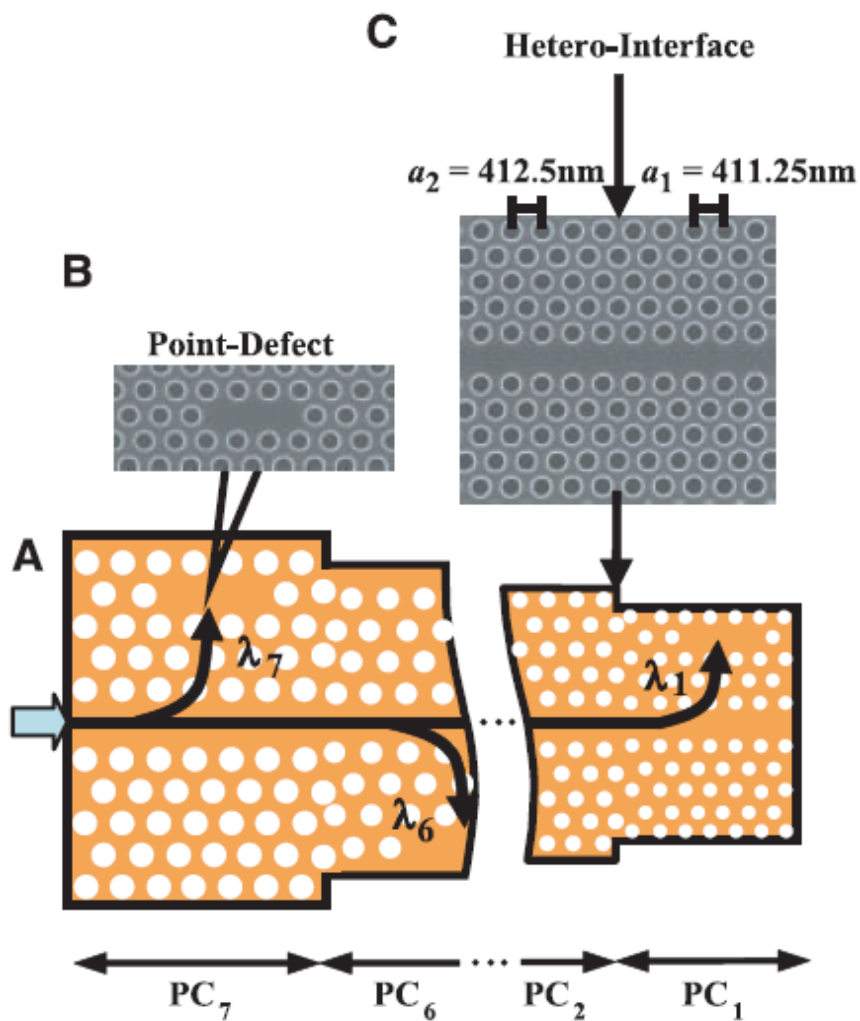
electric field intensity profile

Low-Pumping-Current Lasers



These low threshold current, highly efficient lasers might function as *single photon sources*, which are useful for *cavity quantum electrodynamics* and *quantum information processing*.

Wavelength Channel Drop Devices



B.-S. Song, S. Noda, T. Asano, *Science* **2003**, *300*, 1537.

Chapter Two: Wave Nature of Light

Light waves in a homogeneous medium

Plane electromagnetic waves

Maxwell's wave equation and diverging waves

Refractive index

Group velocity and group index

Snell's law and total internal reflection (TIR)

Fresnel's equations

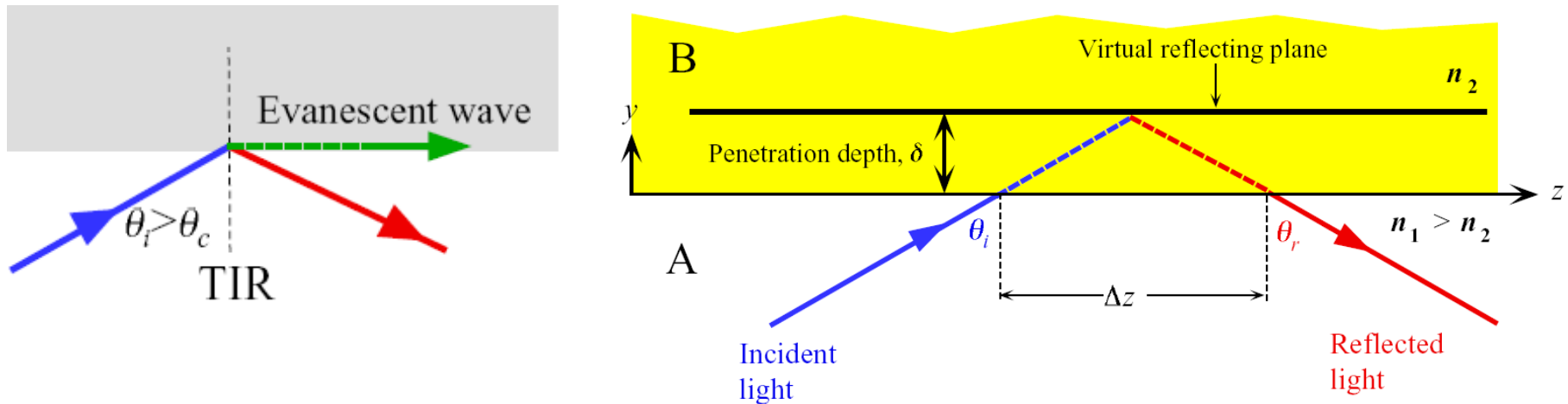
Goos-Hänchen shift and optical tunneling

Diffraction

Fraunhofer diffraction

Diffraction grating

Goos-Hänchen Shift



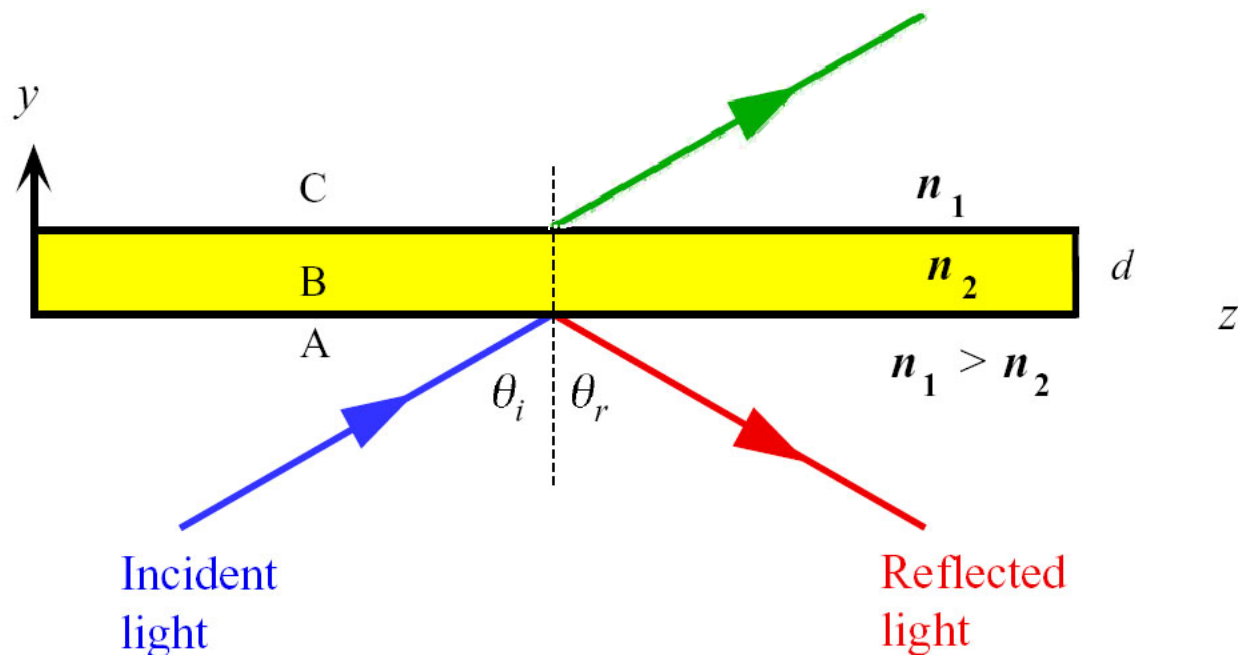
Careful experiments show that the reflected wave at TIR appears to be laterally shifted from the incidence point at the interface. This lateral shift is known as the **Goos-Hänchen shift**.

The reflected wave appears to be reflected from a *virtual plane inside the optically less dense medium*. The spacing between the interface and the virtual plane is **approximately** the penetration depth of the evanescent wave δ when n_1 is very close to n_2 .

$\lambda = 1 \mu\text{m}$
 $n_1 = 1.45$
 $n_2 = 1.43$
 $\theta_i = 85^\circ$
 $\delta = 0.78 \mu\text{m}$
 $\Delta z \approx 18 \mu\text{m}$

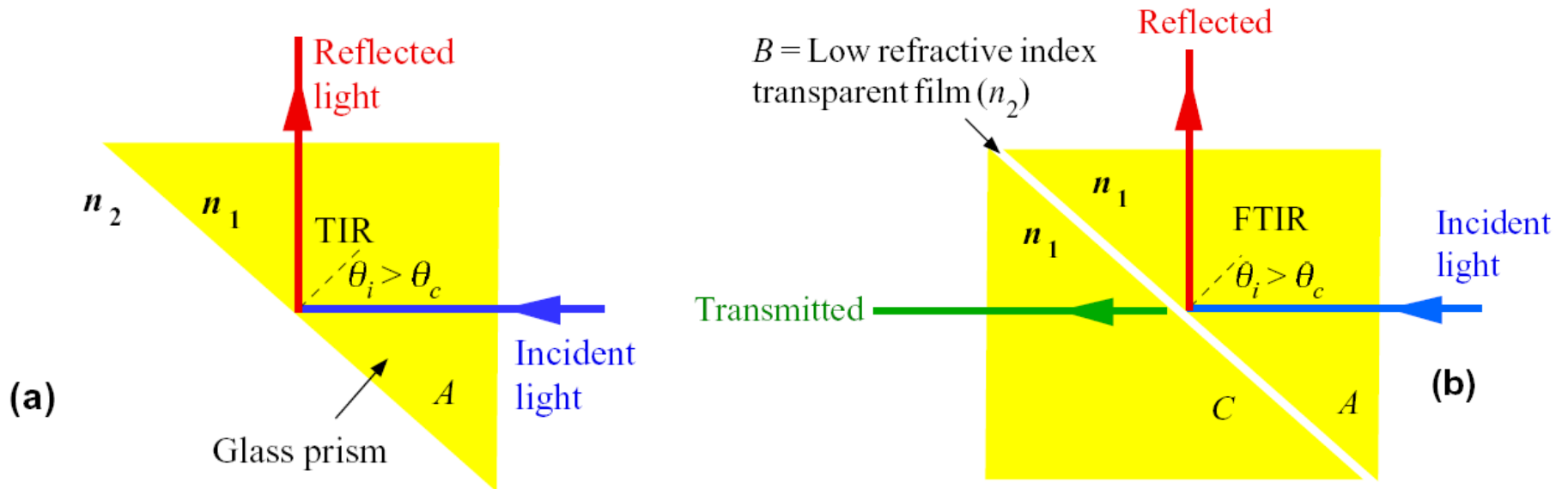
$$\Delta z = 2\delta \tan \theta_i$$

Optical Tunneling

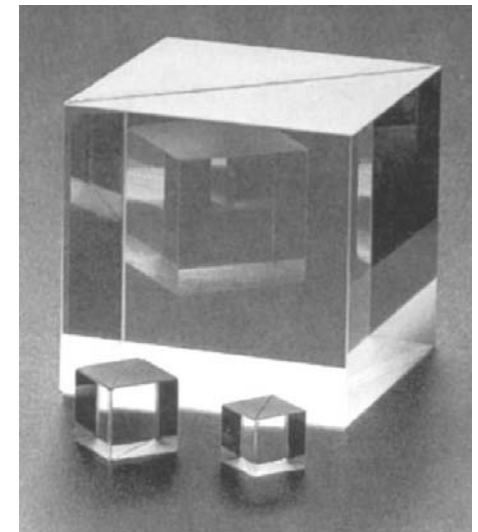


At TIR, we shrink the thickness d of the medium B. When B is sufficiently thin, an attenuated beam emerges on the other side of B in C. This phenomenon, which is forbidden by simple geometrical optics, is called **optical tunneling** or **frustrated total internal reflection**. Optical tunneling is due to the evanescent wave, the field of which penetrates into medium B and reaches the interface BC.

Beam Splitters Based on Frustrated TIR



The extent of light intensity division between the two beams depends on the thickness of the thin layer B and its refractive index.



Chapter Two: Wave Nature of Light

Light waves in a homogeneous medium

Plane electromagnetic waves

Maxwell's wave equation and diverging waves

Refractive index

Group velocity and group index

Snell's law and total internal reflection (TIR)

Fresnel's equations

Goos-Hänchen shift and optical tunneling

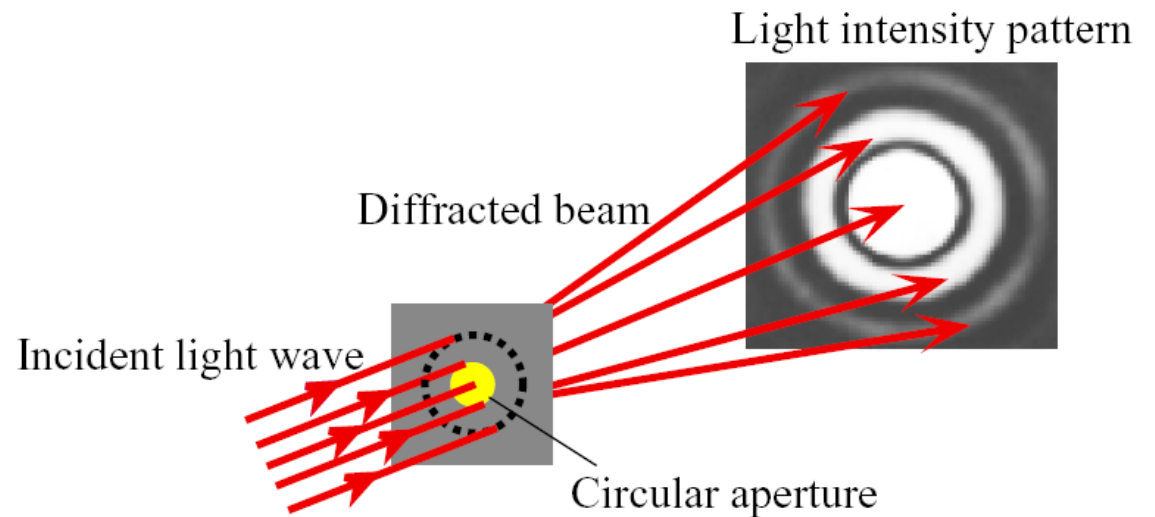
Diffraction

Fraunhofer diffraction

Diffraction grating

Diffraction

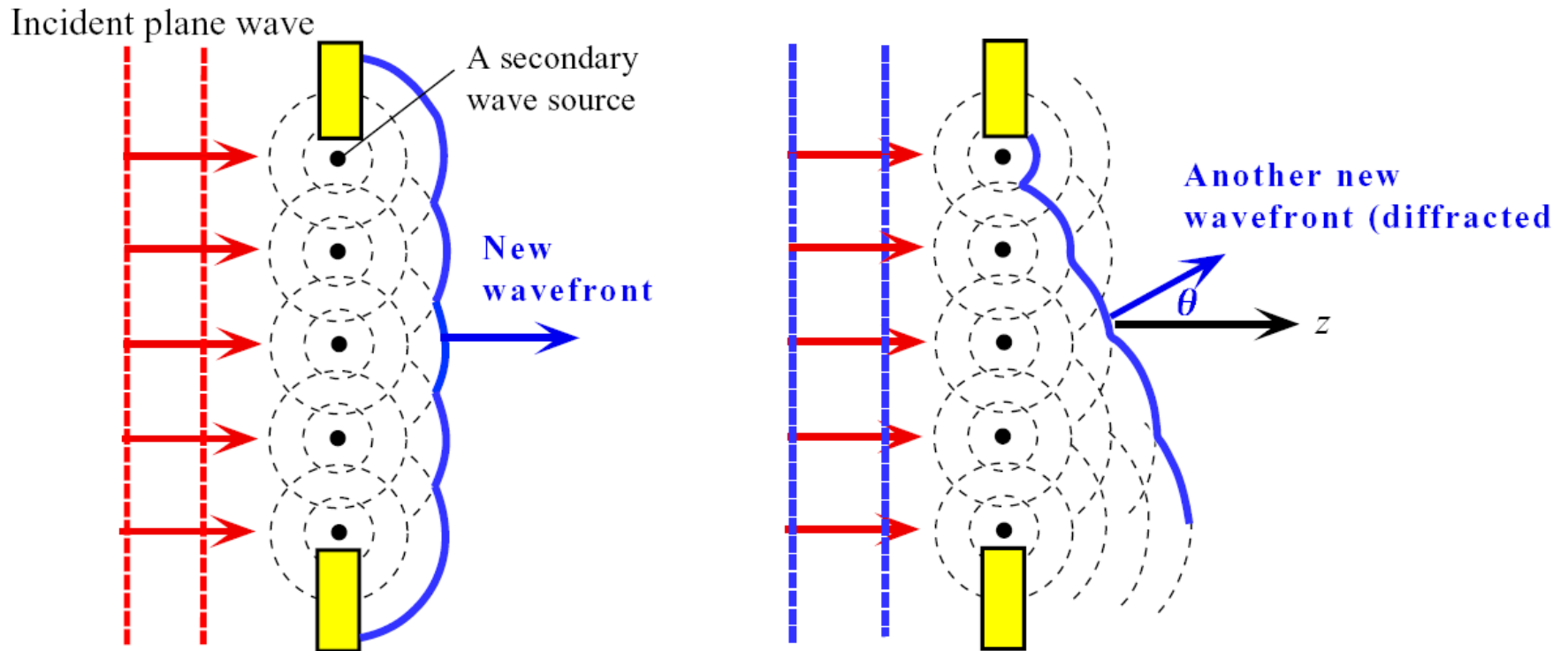
Diffraction occurs when light waves encounter small objects or apertures. For example, the beam passed through a circular aperture is divergent and exhibits an intensity pattern (called **diffraction pattern**) that has bright and dark rings (called **Airy rings**).



Two types of diffraction: (1) In **Fraunhofer diffraction**, the incident light is a plane wave. The observation or detection screen is far away from the aperture so that the received waves also look like plane waves. (2) In **Fresnel diffraction**, the incident and received light waves have significant wavefront curvatures. Fraunhofer diffraction is by far the most important.

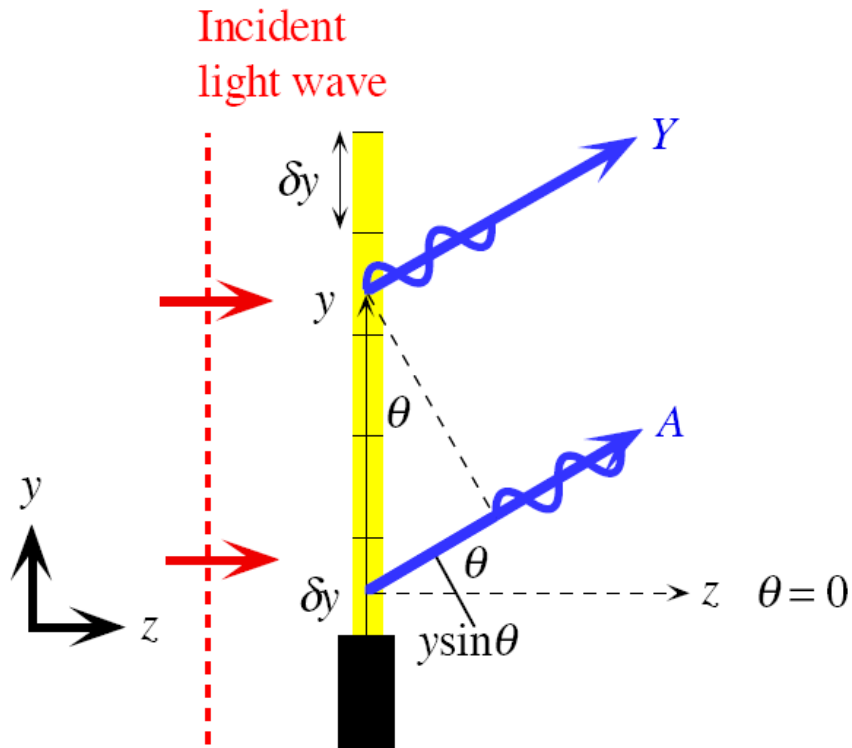
Diffraction

Diffraction can be understood using the **Huygens-Fresnel principle**. Every unobstructed point of a wavefront, at a given instant in time, serves as a source of spherical secondary waves (with the same frequency as that of the primary wave). The amplitude of the optical field at any point beyond is the superposition of all these wavelets (considering their amplitudes and relative phases).



Diffraction Pattern through a 1D Slit

The one-dimensional (1D) slit has a width of a . Divide the width a into N (a large number) coherent sources δy ($= a/N$). Because the slit is uniformly illuminated, the amplitude of each source will be proportional to δy . In the forward direction ($\theta = 0^\circ$), they will be in phase and constitute a forward wave along the z direction. They can also be in phase at some other angles and therefore give rise to a diffracted wave along such directions.



The intensity of the received wave at a point on the screen will be the sum of all waves arriving from all the sources in the slit. The wave Y is out of phase with respect to A by $k y \sin \theta$.

$$\delta E \propto (\delta y) \exp(-jky \sin \theta)$$

$$E(\theta) = C \int_0^a \delta y \exp(-jky \sin \theta)$$

Let $x = y - a/2$, we obtain

$$\begin{aligned} E(\theta) &= C \int_0^a \delta y \exp(-jky \sin \theta) \\ &= C \int_{-a/2}^{a/2} \exp[-jk(x + a/2)\sin \theta] dx \\ &= C e^{-j\frac{1}{2}ka \sin \theta} \int_{-a/2}^{a/2} e^{-jkx \sin \theta} dx \end{aligned}$$

$$\begin{aligned} \int_{-a/2}^{a/2} e^{-jkx \sin \theta} dx &= \frac{1}{-jk \sin \theta} \left(e^{-jk\frac{a}{2} \sin \theta} - e^{jk\frac{a}{2} \sin \theta} \right) \\ &= \frac{-2j \sin\left(\frac{1}{2}ka \sin \theta\right)}{-jk \sin \theta} = \frac{a \sin\left(\frac{1}{2}ka \sin \theta\right)}{\frac{1}{2}ka \sin \theta} \end{aligned}$$

$$E(\theta) = \frac{C e^{-j\frac{1}{2}ka\sin\theta} a \sin\left(\frac{1}{2}ka\sin\theta\right)}{\frac{1}{2}ka\sin\theta}$$

$$I(\theta) = |E(\theta)|^2 = \left[\frac{Ca \sin\left(\frac{1}{2}ka\sin\theta\right)}{\frac{1}{2}ka\sin\theta} \right]^2$$

The zero intensity occurs when:

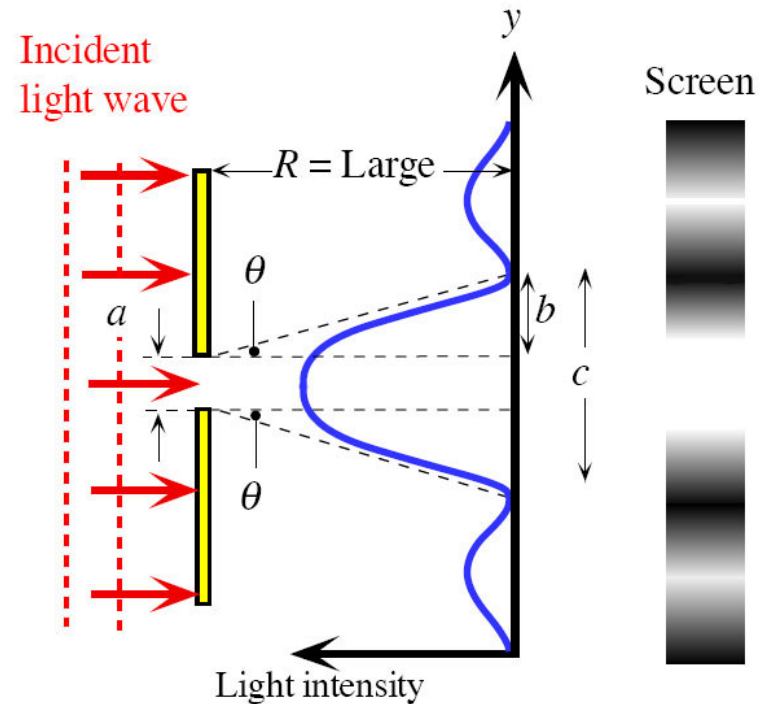
$$\frac{1}{2}ka\sin\theta = \frac{\pi}{\lambda}a\sin\theta = m\pi$$

$$\sin\theta = \frac{m\lambda}{a} \quad m = \pm 1, \pm 2, \dots$$

$$\lambda = 1.3 \mu\text{m}$$

$$a = 100 \mu\text{m}$$

$$\text{divergence } 2\theta = 1.5^\circ$$



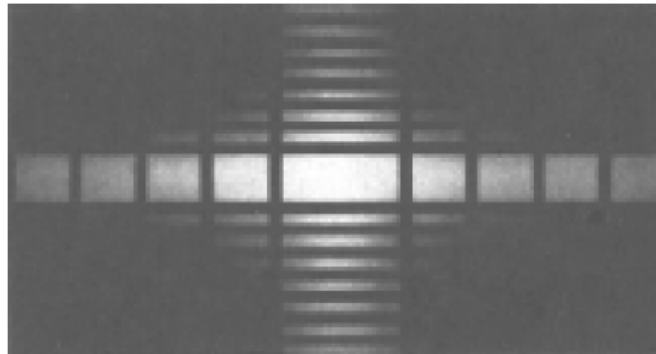
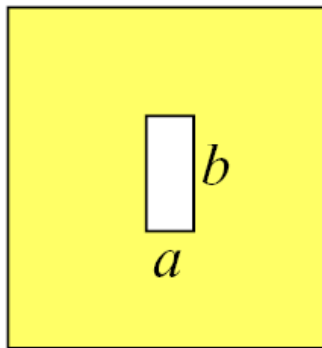
- **Bright and dark regions**
- **The central bright region is larger than the slit width (the transmitted beam is diverging)**

The diffraction patterns from two-dimensional (2D) apertures such as rectangular and circular apertures are more complicated to calculate but they use the same principle based on the interference of waves emitted from all point sources in the aperture.

The diffraction pattern from a circular aperture of diameter D has a central bright region. The divergence of this bright region is determined by

$$\sin \theta = 1.22 \frac{\lambda}{D}$$

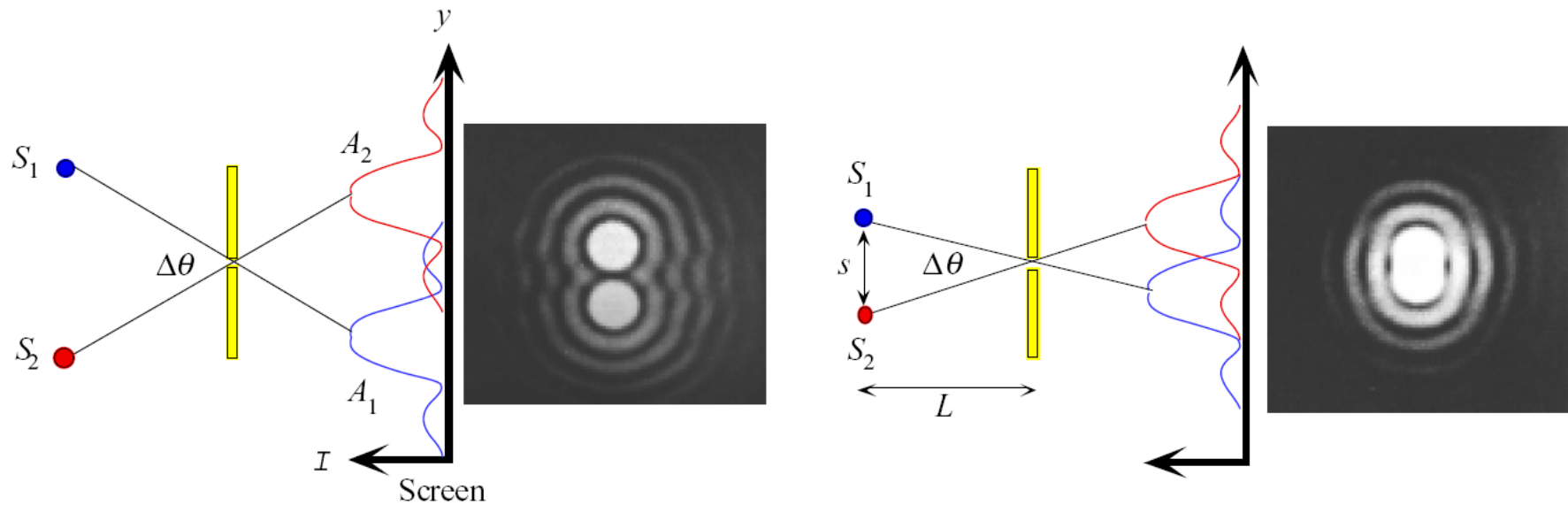
Diffraction from rectangular apertures



Why is the central bright region rotated relative to the rectangular aperture?

Diffraction-Limited Resolution

Consider two neighboring coherent sources with an angular separation of $\Delta\theta$ examined through an imaging system with an aperture of diameter D . The aperture produces a diffraction pattern of the sources. As the sources get closer, their diffraction patterns overlap more.



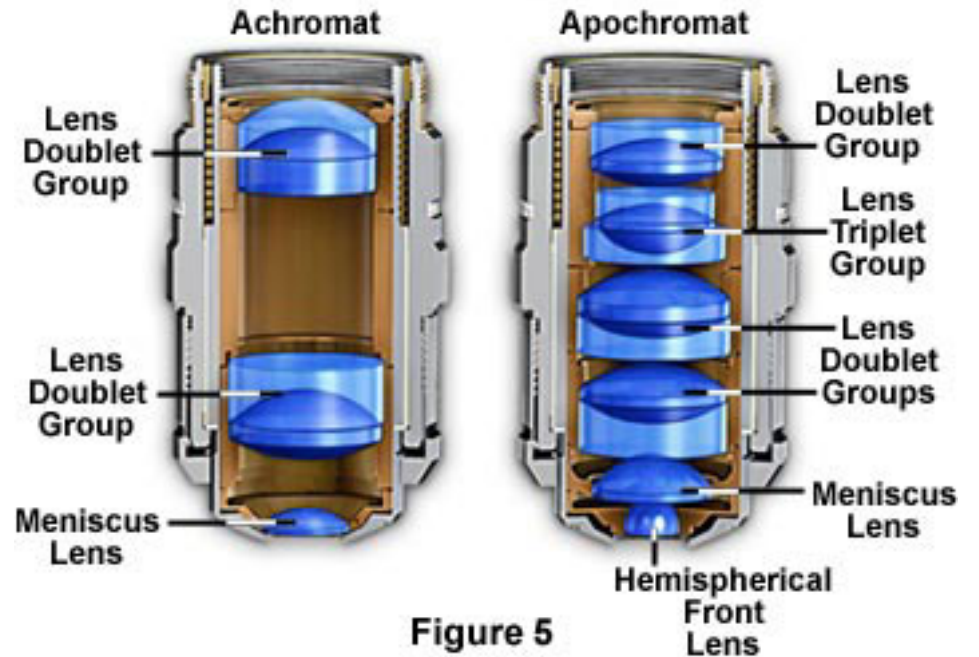
According to **Rayleigh criterion**, the two spots are just resolvable when the principle maximum of one diffraction pattern coincides with the minimum of the other, which is given by the condition:

$$\sin(\Delta\theta_{\min}) = 1.22 \frac{\lambda}{D}$$

Diffraction-Limited Resolution



Achromatic and Apochromatic Objective Correction



Objectives are key components in an optical microscope.

Diffraction-Limited Resolution

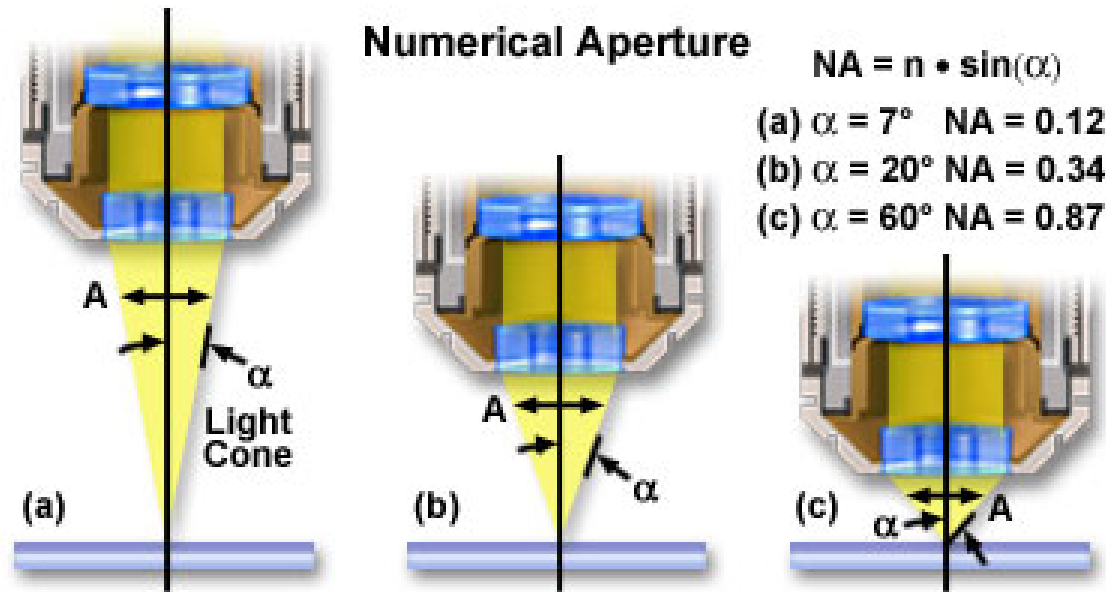


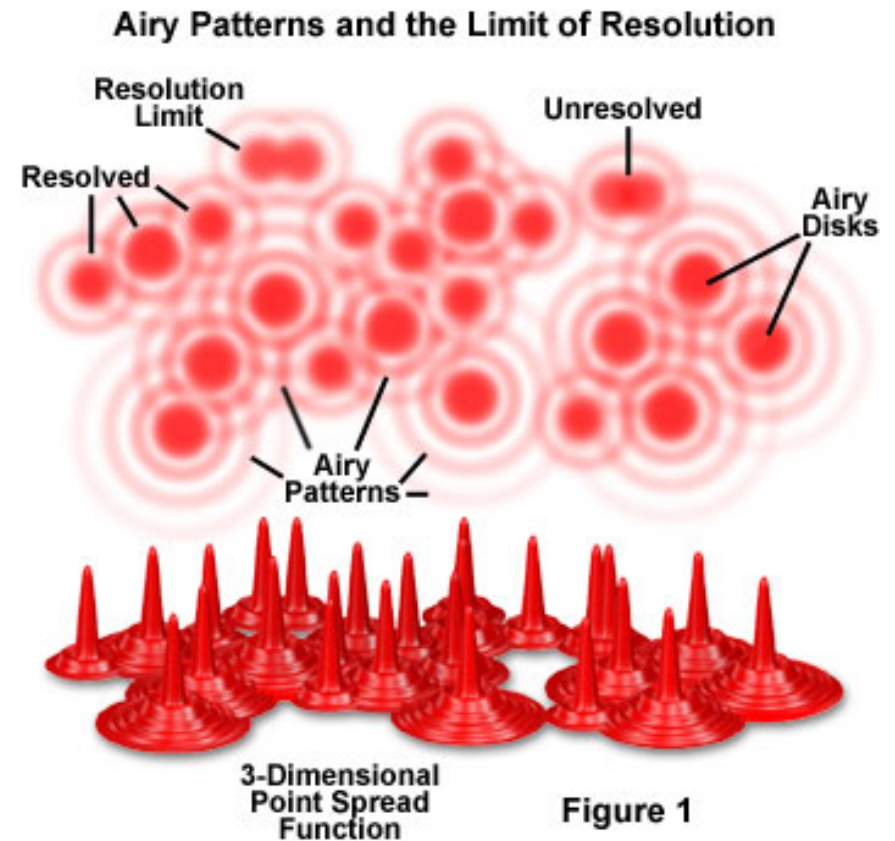
Figure 1

$$NA = n \cdot \sin(\alpha)$$

The numerical aperture (NA) of a microscope objective is a measure of its ability to gather light and resolve fine specimen detail at a fixed object distance.

Diffraction-Limited Resolution

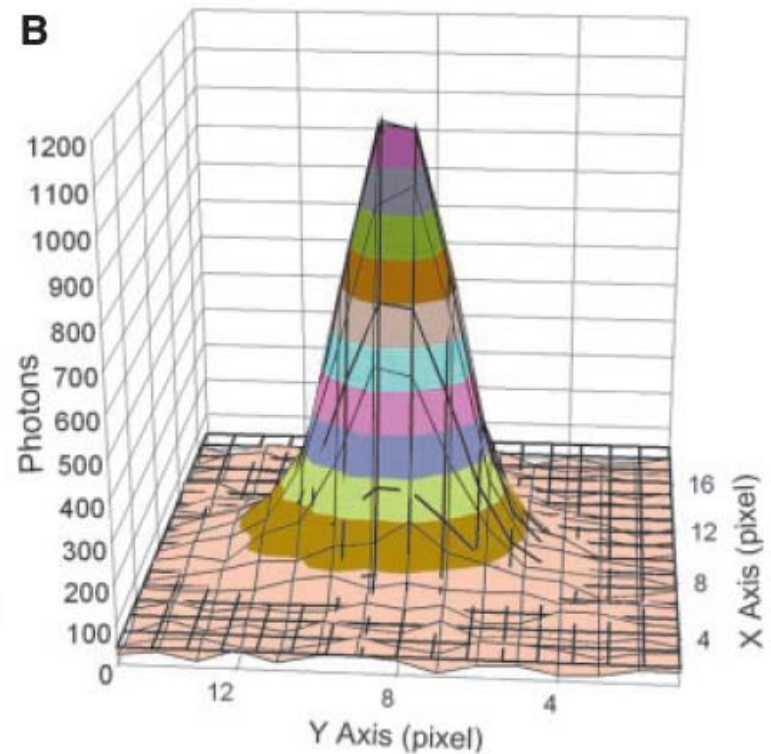
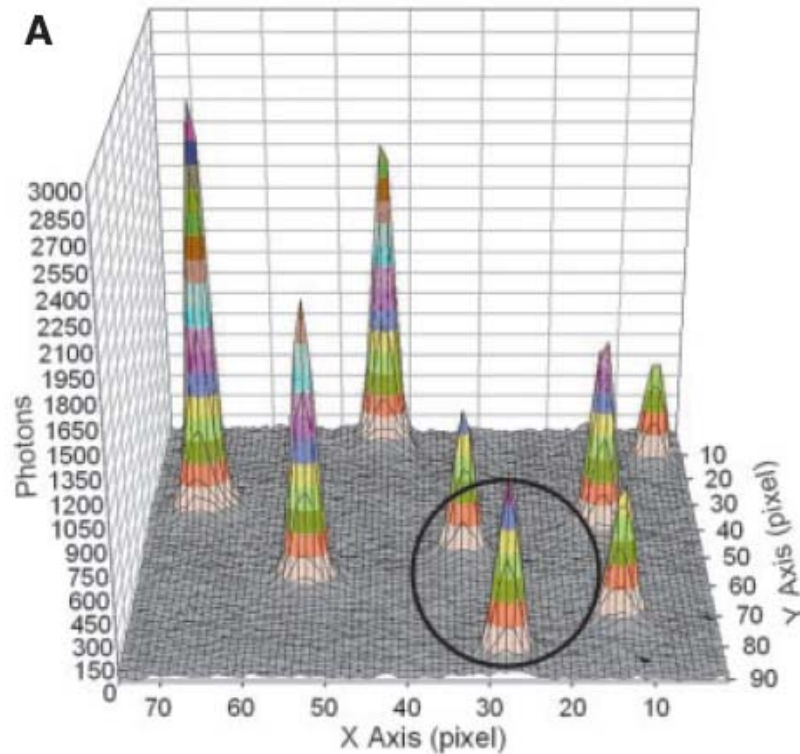
$$\text{resolution } r = \frac{0.61\lambda}{\text{NA}}$$



The resolution of an optical microscope is defined as the shortest distance between two points on a specimen that can still be distinguished by the observer or camera system as separate entities. It is the most important feature of the optical system and influences the ability to distinguish between fine details of a particular specimen.

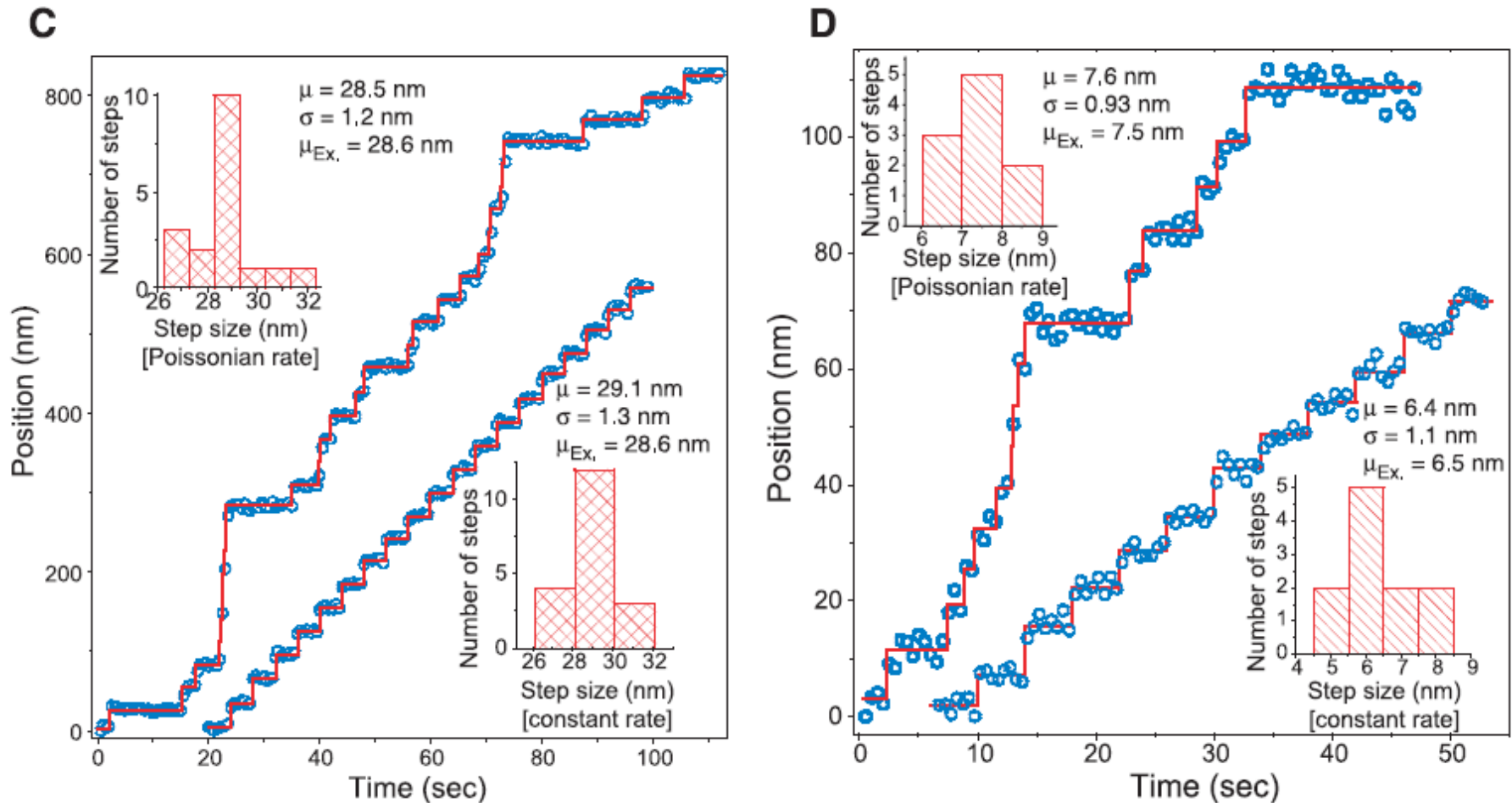
Overcome Diffraction-Limited Resolution

If a sufficient number of photons are collected from a single fluorescent molecule, the center of this molecule can be determined up to **1.5-nm resolution** by curve-fitting the fluorescence image to a Gaussian function.



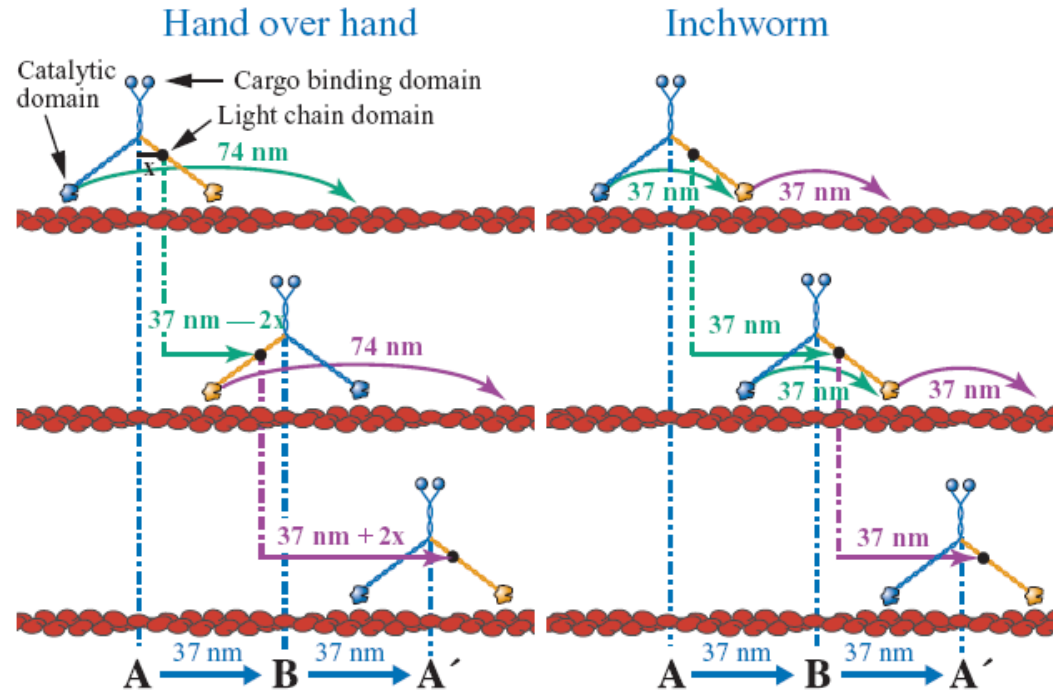
A. Yildiz, J. N. Forkey, S. A. McKinney, T. Ha, Y. E. Goldman, P. R. Selvin, *Science* **2003**, 300, 2061.

Overcome Diffraction-Limited Resolution



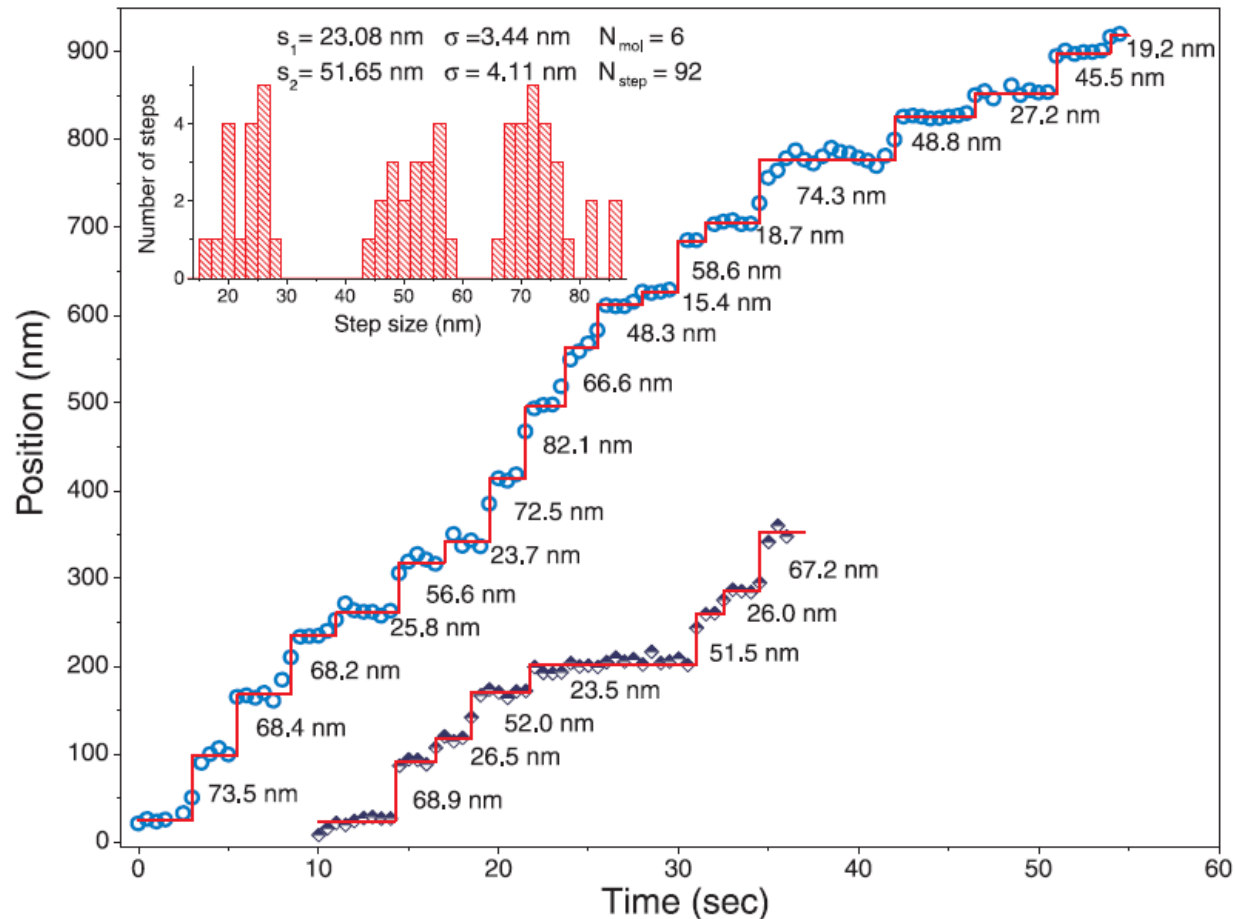
Discernable 30-nm and 7-nm steps are readily observed upon moving the cover slip, either at a constant rate or a Poisson-distributed rate.

Overcome Diffraction-Limited Resolution



Myosin (肌球蛋白, 肌凝蛋白) V is a cargo-carrying motor that takes 37-nm center of mass steps along actin (肌动蛋白) filaments. It has two heads held together by a coiled-coil stalk. Hand-over-hand and inchworm models have been proposed for its movement along actin filaments. For a single fluorescent molecule attached to the light chain domain of myosin V, the inchworm model predicts a uniform step size of 37 nm, whereas the hand-over-hand model predicts alternating steps of $37 - 2x$, $37 + 2x$, where x is the in-plane distance of the dye from the midpoint of the myosin.

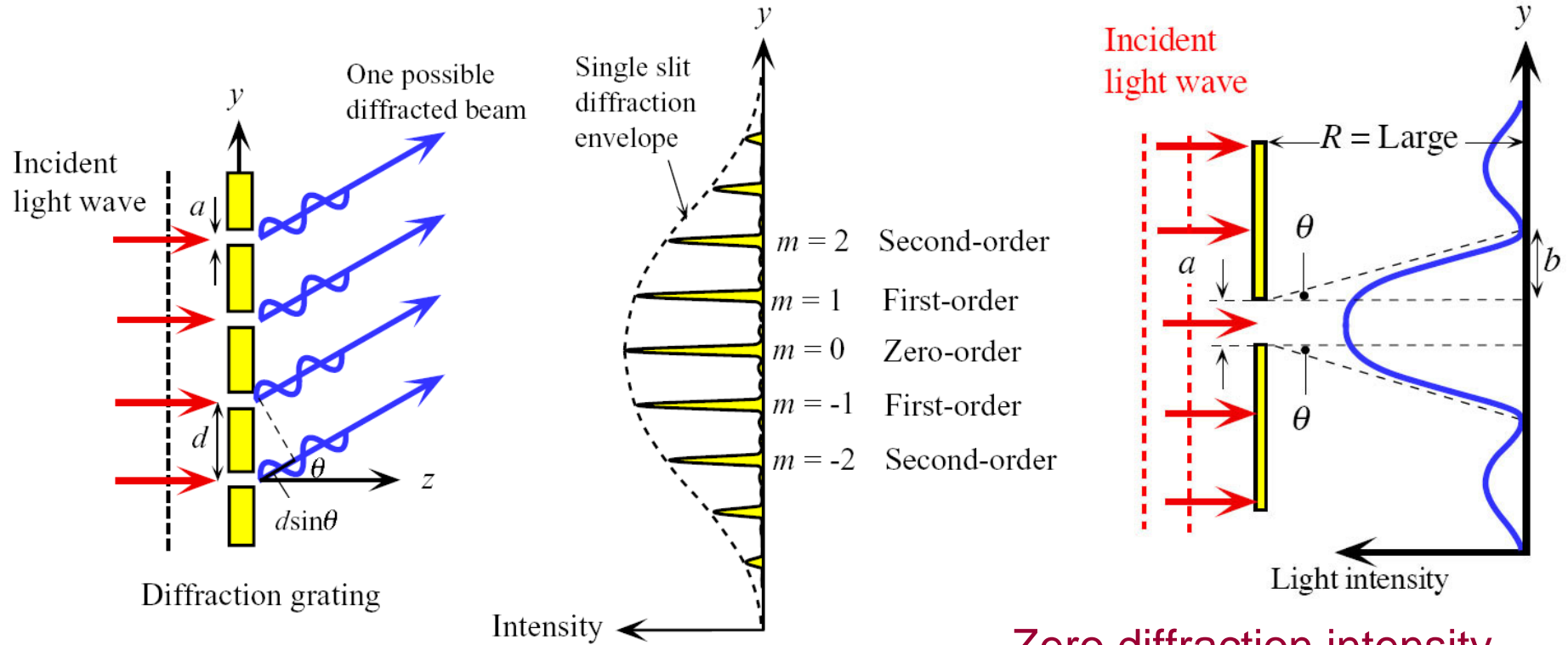
Overcome Diffraction-Limited Resolution



The dye is 7 nm from the center of mass along the direction of motion. This result indicates that myosin V moves in a hand-over-hand fashion along actin filaments.

Diffraction Grating

A **diffraction grating** in its simplest form has a periodic series of slits. An incident beam of light is diffracted in certain well-defined directions that depend on the wavelength and the grating periodicities.



Bragg diffraction condition

(maximum intensity):

$$\sin \theta = m \frac{\lambda}{d} \quad m = \pm 1, \pm 2, \dots$$

Zero diffraction intensity condition for a single slit:

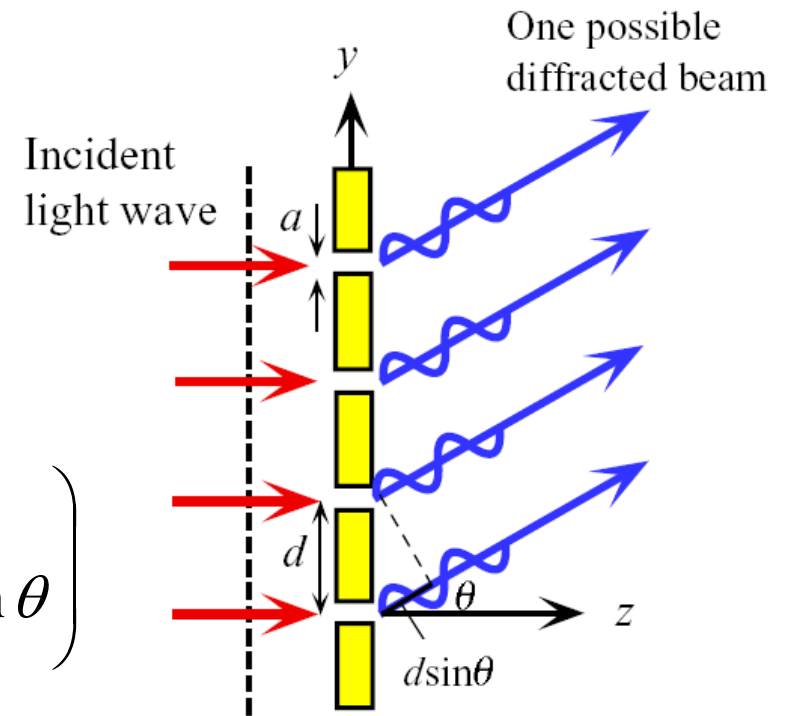
$$\sin \theta = \frac{m\lambda}{a} \quad m = \pm 1, \pm 2, \dots$$

Let's try to derive the actual diffraction intensity from the grating by assuming a plane incident wave and that there are N slits.

For the diffraction from a single slit:

$$E_{\text{slit}}(\theta) = \frac{C e^{-j\frac{1}{2}ka\sin\theta} a \sin\left(\frac{1}{2}ka\sin\theta\right)}{\frac{1}{2}ka\sin\theta}$$

$$= \frac{C_1 e^{-j\alpha} \sin\alpha}{\alpha} \quad \left(\begin{array}{l} C_1 = Ca \\ \alpha = \frac{1}{2}ka\sin\theta \end{array} \right)$$



For the diffraction from the grating:

$$E_{\text{grating}}(\theta) = E_{\text{slit}}(\theta) + E_{\text{slit}}(\theta)e^{-jkd\sin\theta} + E_{\text{slit}}(\theta)e^{-j2kd\sin\theta} + \dots + E_{\text{slit}}(\theta)e^{-j(N-1)kd\sin\theta}$$

$$= E_{\text{slit}}(\theta) \left[1 + e^{-jkd\sin\theta} + e^{-j2kd\sin\theta} + \dots + e^{-j(N-1)kd\sin\theta} \right]$$

A geometrical progression

$$E_{\text{grating}}(\theta) = E_{\text{slit}}(\theta) \frac{1 - e^{-jNkd \sin \theta}}{1 - e^{-jkd \sin \theta}}$$

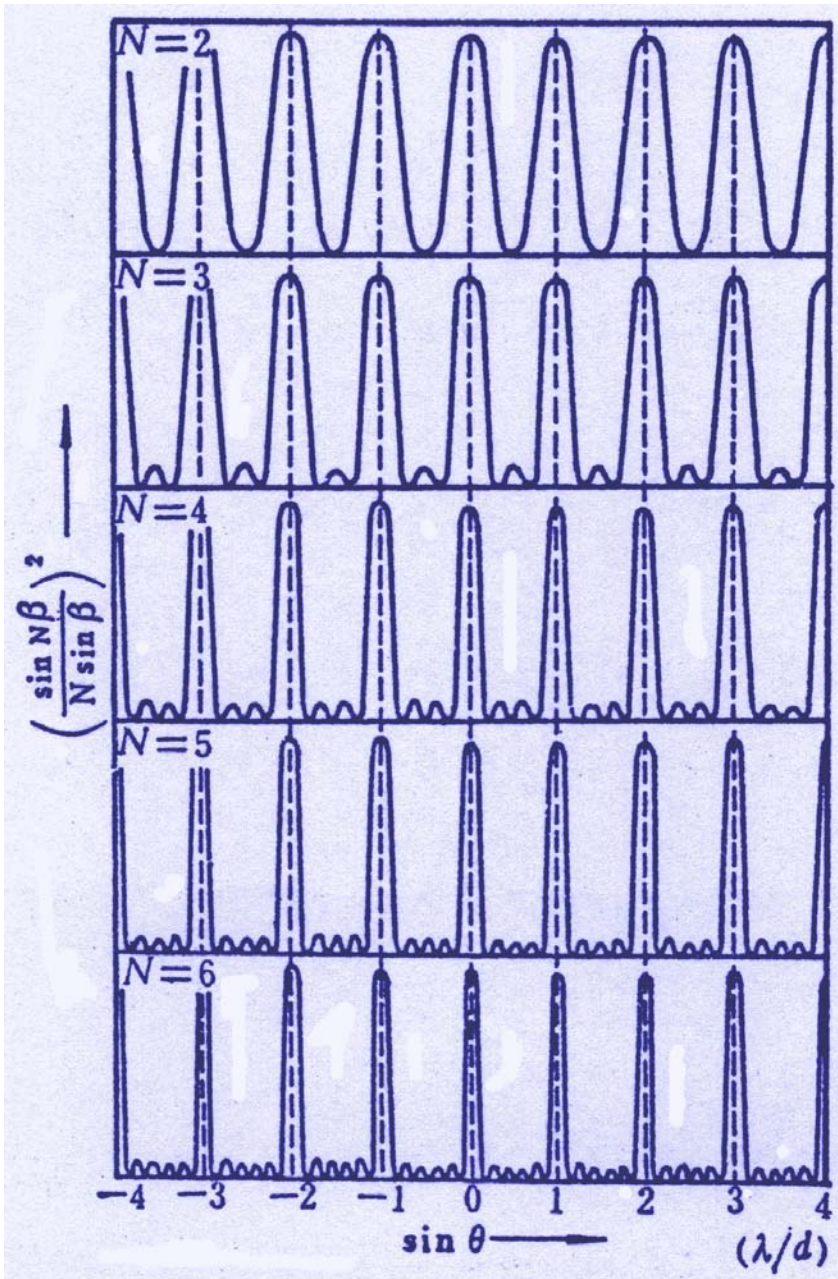
$$I_{\text{grating}}(\theta) = |E_{\text{grating}}(\theta)|^2 = C_1^2 \left(\frac{\sin \alpha}{\alpha} \right)^2 \left| \frac{1 - e^{-jNkd \sin \theta}}{1 - e^{-jkd \sin \theta}} \right|^2$$

$$\left| \frac{1 - e^{-jNkd \sin \theta}}{1 - e^{-jkd \sin \theta}} \right|^2 = \frac{|1 - \cos(Nkd \sin \theta) + j \sin(Nkd \sin \theta)|^2}{|1 - \cos(kd \sin \theta) + j \sin(kd \sin \theta)|^2} = \frac{2 - 2 \cos(Nkd \sin \theta)}{2 - 2 \cos(kd \sin \theta)}$$

$$= \frac{1 - \cos(Nkd \sin \theta)}{1 - \cos(kd \sin \theta)} = \frac{2 \sin^2 \left(\frac{1}{2} Nkd \sin \theta \right)}{2 \sin^2 \left(\frac{1}{2} kd \sin \theta \right)} = \left(\frac{\sin N\beta}{\sin \beta} \right)^2$$

$$\left(\beta = \frac{1}{2} kd \sin \theta \right)$$

$$I_{\text{grating}}(\theta) = C_1^2 \left(\frac{\sin \alpha}{\alpha} \right)^2 \left(\frac{\sin N\beta}{\sin \beta} \right)^2$$



When $\begin{cases} \sin N\beta = 0 \\ \sin \beta = 0 \end{cases} \left(\beta = \frac{1}{2}kd \sin \theta \right)$

$\left(\frac{\sin N\beta}{\sin \beta} \right)^2 = N^2$ is maximum.



$\beta = m\pi$
 $\sin \theta = m \frac{\lambda}{d} \quad m = \pm 1, \pm 2, \dots$

When $\begin{cases} \sin N\beta = 0 \\ \sin \beta \neq 0 \end{cases} \left(\frac{\sin N\beta}{\sin \beta} \right)^2 = 0$

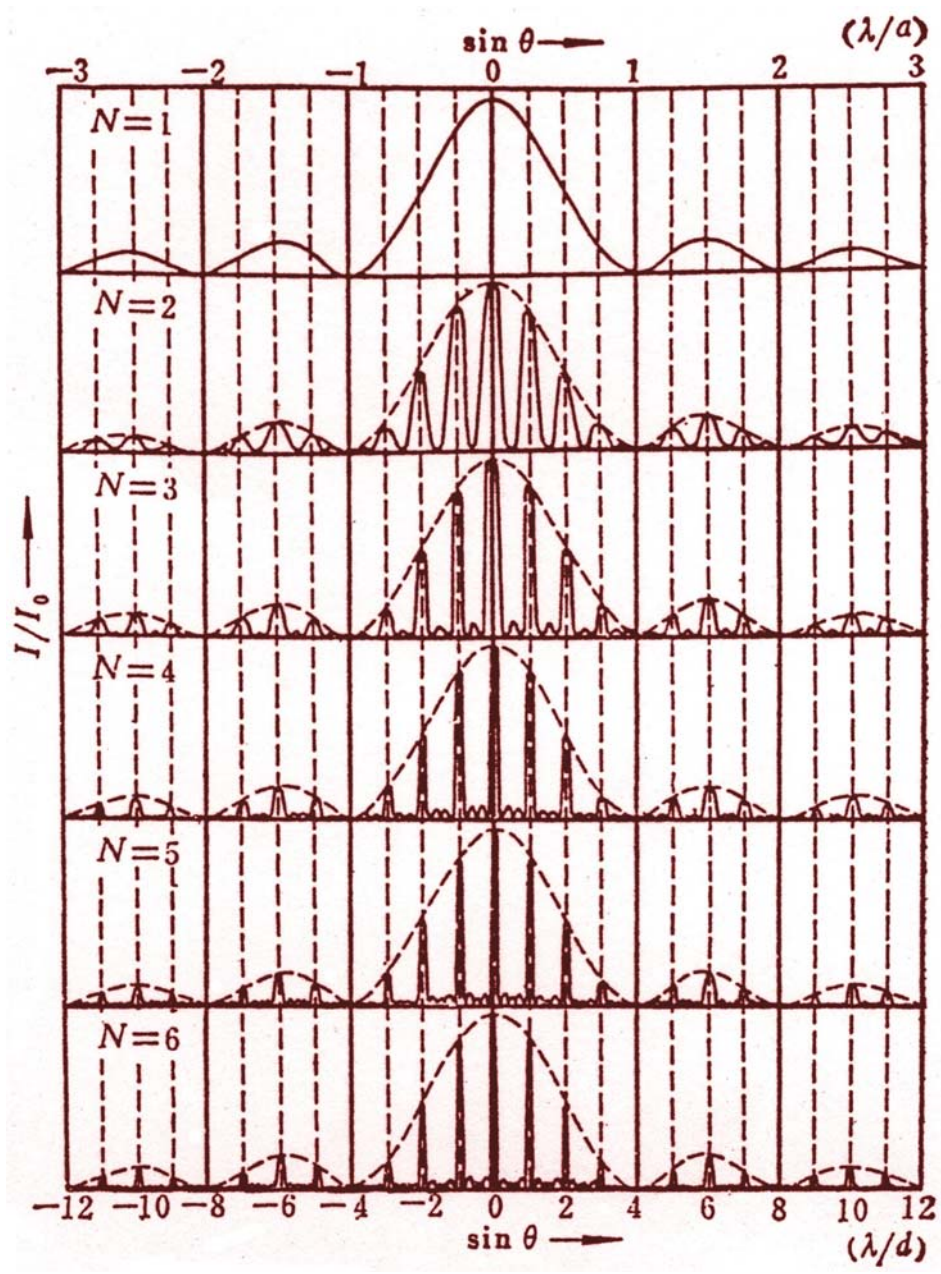
$\sin \theta = \left(m + \frac{k}{N} \right) \frac{\lambda}{d} \quad \begin{cases} m = \pm 1, \pm 2, \dots \\ k = 1, 2, \dots, N-1 \end{cases}$

$N-1$ zero points, $N-2$ maxima

$$I_{\text{grating}}(\theta) = C_1^2 \left(\frac{\sin \alpha}{\alpha} \right)^2 \left(\frac{\sin N\beta}{\sin \beta} \right)^2$$

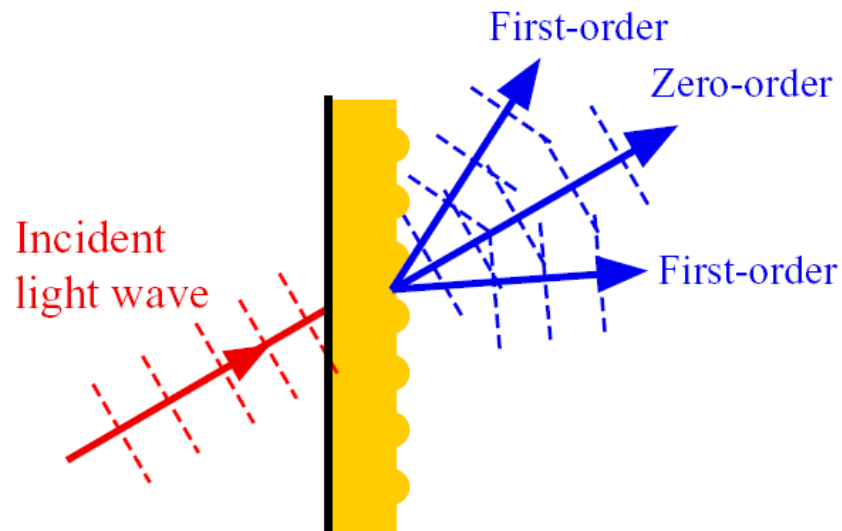
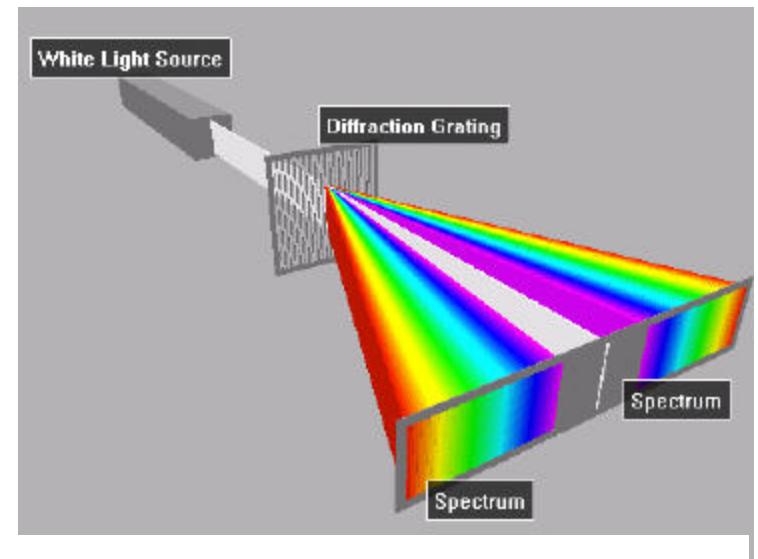
$$\begin{cases} \alpha = \frac{1}{2} ka \sin \theta \\ \beta = \frac{1}{2} kd \sin \theta \end{cases}$$

The amplitude of the diffracted beam is modulated by the diffraction amplitude of a single slit since the latter is spread substantially than the former.



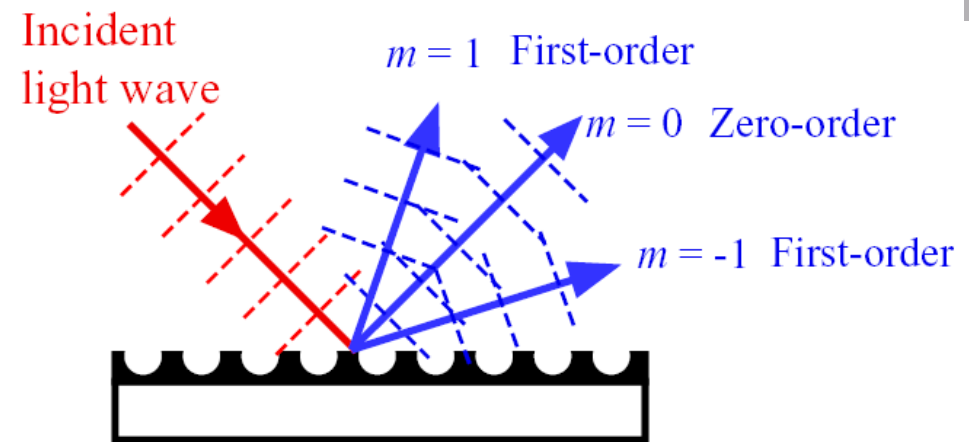
The diffraction grating provides a means of deflecting an incoming light by an amount that depends on its wavelength. This is of great use in spectroscopy.

In reality, any periodic variations in the refractive index would serve as a diffraction grating.



(a) Transmission grating

$$\sin \theta_m - \sin \theta_i = m \frac{\lambda}{d}$$



(b) Reflection grating

$$m = \pm 1, \pm 2, \dots$$

$$\sin \theta_m + \sin \theta_i = m \frac{\lambda}{d}$$

Reading Materials

S. O. Kasap, “Optoelectronics and Photonics: Principles and Practices”, Prentice Hall, Upper Saddle River, NJ 07458, 2001, Chapter 1, “Wave Nature of Light”.