7. Differentiation of Trigonometric Function

RADIAN MEASURE. Let s denote the length of arc AB intercepted by the central angle AOB on a circle of radius r and let S denote the area of the sector AOB. (If s is 1/360 of the circumference, $\angle AOB = 1^0$; if $s = r$, $\angle AOB$ = 1 radian). Suppose $\angle AOB$ is measured as a degrees; then

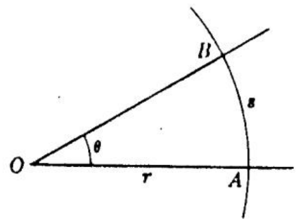


Fig. 12-1

(i) $\int \csc^2 u\, du = -\cot u + C$ and

$$S = \frac{\pi}{360} \alpha\, r^2$$

Suppose next that $\angle AOB$ is measured as θ radian; then

(ii) $s = \theta r$ and $S = ½\, \theta r^2$

A comparison of (i) and (ii) will make clear one of the advantages of radian measure.

TRIGONOMETRIC FUNCTIONS. Let θ be any real number. Construct the angle whose measure is θ radians with vertex at the origin of a rectangular coordinate system and initial side along the positive x-axis. Take $P(x, y)$ on the terminal side of the angle a unit distance from O; then $\sin\theta = y$ and $\cos\theta = x$. The domain of definition of both $\sin\theta$ and $\cos\theta$ is the set or real number; the range of $\sin\theta$ is $-1 \le y \le 1$ and the range of $\cos\theta$ is $-1 \le x \le 1$. From

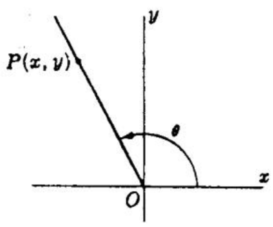


Fig. 12-2

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{and} \quad \sec\theta = \frac{1}{\cos\theta}$$

it follows that the range of both $\tan\theta$ and $\sec\theta$ is set of real numbers while the domain of definition ($\cos\theta \neq 0$) is $\theta \neq \pm\frac{2n-1}{2}\pi$, (n = 1, 2, 3, …). It is left as an exercise for the reader to consider the functions $\cot\theta$ and $\csc\theta$.

In problem 1, we prove

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

(Had the angle been measured in degrees, the limit would have been $\pi/180$. For this reason, *radian measure is always used in the calculus*)

RULES OF DIFFERENTIATION. Let *u* be a differentiable function of *x;* then

14. $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$

15. $\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$

16. $\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$

17. $\frac{d}{dx}(\cot u) = -\csc^2 u \frac{du}{dx}$

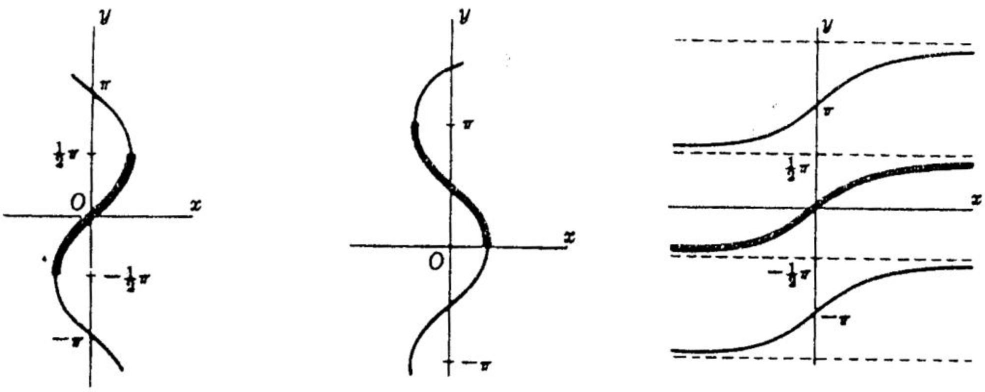
18. $\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}$

19. $\frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}$

8. Differentiation of Inverse trigonometric functions

THE INVERSE TRIGONOMETRIC FUNCTIONS. If $x = \sin y$, the inverse function is written $y = arc \sin x$. The domain of definition of arc sin *x* is *–1 ≤x ≤1,* the range of sin *y;* the range of arc sin *x* is the set if real numbers, the domain of definition of sin *y*. The domain if definition and the range of the remaining inverse trigonometric functions may be established in a similar manner.

The inverse trigonometric functions are multi-valued. In order that there be agreement on separating the graph into single-valued arcs, we define below one such arc (called the *principal branch*) for each function. In the accompanying graphs, the principal branch is indicated by a thickening of the line.



y = arc sin *x*   *y* = arc cos *x*   *y* = arc tan *x*

Fig. 13-1

Function	Principal Branch
y = arc sin *x*	$-\frac{1}{2}\pi \leqq y \leqq \frac{1}{2}\pi$
y = arc cos *x*	$0 \leqq y \leqq \pi$
y = arc tan *x*	$-\frac{1}{2}\pi < y < \frac{1}{2}\pi$
y = arc cot *x*	$0 < y < \pi$
y = arc sec *x*	$-\pi \leqq y < -\frac{1}{2}\pi$, $0 \leqq y < \frac{1}{2}\pi$
y = arc csc *x*	$-\pi < y \leqq -\frac{1}{2}\pi$, $0 < y \leqq \frac{1}{2}\pi$

RULES OF DIFFERENTIATION. Let *u* be a differentiable function of *x*, then

20. $\frac{d}{dx}(\text{arc sin u}) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

21. $\frac{d}{dx}(\text{arc cos u}) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

22. $\frac{d}{dx}(\text{arc tan u}) = \frac{1}{1+u^2}\frac{du}{dx}$

23. $\frac{d}{dx}(\text{arc cot u}) = -\frac{1}{1+u^2}\frac{du}{dx}$

24. $\frac{d}{dx}(\text{arc sec u}) = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$

25. $\frac{d}{dx}(\text{arc csc u}) = -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$

9. DIFFERENTIATION OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

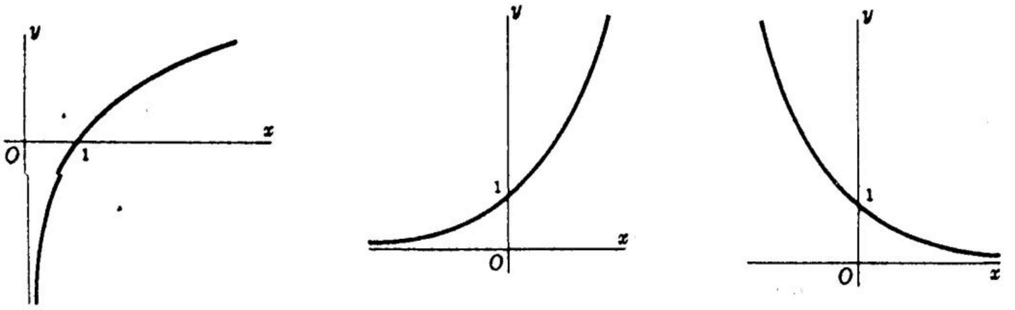
THE NUMBER *e* = $\lim_{h\to+\infty}\left(1+\frac{1}{h}\right)^h = \lim_{k\to 0}(1+k)^{1/k}$

$$= 1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\ldots = 2.178\,28\ldots$$

NOTATION. If $a > 0$ and $a \neq 1$, and if $a^y = x$, then $y = \log_a x$

$$y = \log_e x = \ln x \qquad y = \log_{10} x = \log x$$

The domain of definition is $x > 0$; the range is the set of real numbers.



$y = \ln x$ $\qquad$ $y = e^{ax}$ $\qquad$ $y = e^{-ax}$

Fig. 14-1

Rules of differentiation. If *u* is a differentiable function of *x*,

26. $\frac{d}{dx}({}^a\log u) = \frac{1}{u\ln a}\frac{du}{dx}$, $(a > 0, a \neq 1)$

27. $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$

28. $\frac{d}{dx}(a^u) = a^u \ln a\frac{du}{dx}$, $(a > 0)$

29. $\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$

LOGARITHMIC DIFFERENTIATION. If a differentiable function *y = f(x)* is the product of several factors, the process of differentiation may be simplified by taking the natural logarithm of the function before differentiating or, what is the same thing, by using the formula

30. $\frac{d}{dx}(y) = y\frac{du}{dx}(\ln y)$

10. DIFFERENTIATION OF HYPERBOLIC FUNCTIONS

DEFINITIONS OF HYPERBOLIC FUNCTION. For *u* any real number, except where noted:

$$\sinh u = \frac{e^u - e^{-u}}{2} \qquad \coth u = \frac{1}{\tanh u} = \frac{e^u + e^{-u}}{e^u - e^{-u}},\ (u \neq 0)$$

$$\cosh u = \frac{e^u + e^{-u}}{2} \qquad \operatorname{sech} u = \frac{1}{\cosh u} = \frac{2}{e^u + e^{-u}}$$

$$\tanh u = \frac{\sinh u}{\cosh u} = \frac{e^u - e^{-u}}{e^u + e^{-u}} \qquad \operatorname{csch} u = \frac{1}{\sinh u} = \frac{2}{e^u - e^{-u}},\ (u \neq 0)$$

DIFFERENTIATION FORMULAS. If *u* is a differentiable function of *x*,

31. $\frac{d}{dx}(\sinh u) = \cosh u\frac{du}{dx}$

32. $\frac{d}{dx}(\cosh u) = \sinh u\frac{du}{dx}$

33. $\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u\frac{du}{dx}$

34. $\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u\frac{du}{dx}$

35. $\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u\frac{du}{dx}$

36. $\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u\frac{du}{dx}$

DEFINITIONS OF INVERSE HYPERBOLIC FUNCTIONS.

$$\sinh^{-1} u = \ln(u + \sqrt{1+u^2}), \text{all } u \qquad \coth^{-1} u = \tfrac{1}{2}\ln\frac{u+1}{u-1}, \quad (u^2 > 1)$$

$$\cosh^{-1} u = \ln(u + \sqrt{u^2 - 1}), \quad (u \geqq 1) \qquad \operatorname{sech}^{-1} u = \ln\frac{1+\sqrt{1-u^2}}{u}, \quad (0 < u \leqq 1)$$

$$\tanh^{-1} u = \tfrac{1}{2}\ln\frac{1+u}{1-u}, \quad (u^2 < 1) \qquad \operatorname{csch}^{-1} u = \ln\left(\frac{1}{u} + \frac{\sqrt{1+u^2}}{|u|}\right), \quad (u \neq 0)$$

Differentiation formulas. If *u* is a differentiable function of *x*,

37. $\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$

38. $\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad (u > 1)$

39. $\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad (u^2 < 1)$

40. $\frac{d}{dx}(\coth^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad (u^2 > 1)$

41. $\frac{d}{dx}(\text{sech}^{-1} u) = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad (0 < u < 1)$

42. $\frac{d}{dx}(\text{csch}^{-1} u) = \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \quad (u \neq 0)$

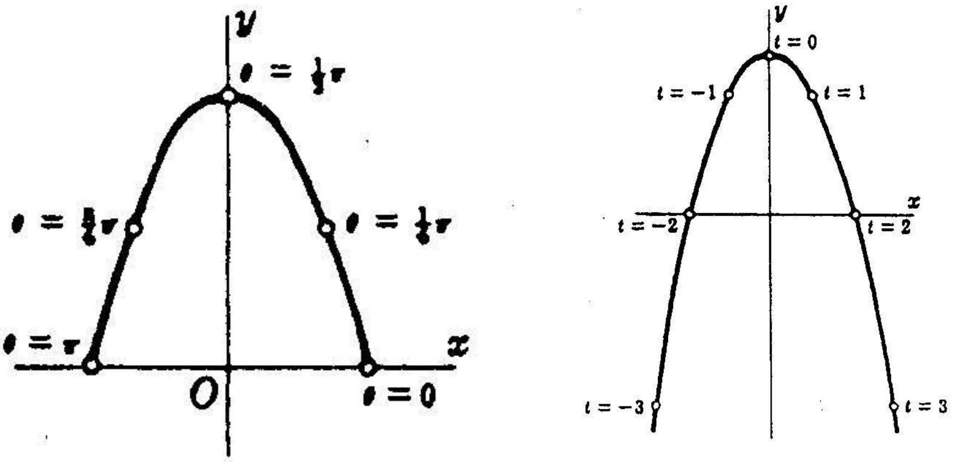
11. PARAMETRIC REPRESENTATION OF CURVES

PARAMETRIC EQUATIONS. If the coordinates (x, y) of a point P on a curve are given as functions $x = f(u)$, $y = g(u)$ of a third variable or *parameters u*, the equations $x = f(u)$, $y = g(u)$ are called *parametric equations* of the curve.

Example:

(a) $x = \cos\theta,\ y = 4\sin^2\theta$ are parametric equations, with parameter θ, of the parabola $4x + y = 4$, since $4x^2 + y = 4\cos^2\theta + 4\sin^2\theta = 4$

(b) $x = \frac{1}{2}t,\ y = 4 - t^2$ is another parametric representation, with parameter t, of the same curve.



(a) (b)

Fig. 16-1

It should be noted that the first set of parametric equations represents only a portion of the parabola, whereas the second represents the entire curve.

THE FIRST DERIVATIVE $\frac{dy}{dx}$ is given by $\frac{dy}{dx} = \frac{dy/du}{dx/du}$

The second derivative $\frac{d^2y}{dx^2}$ is given by $\frac{d^2y}{dx^2} = \frac{d}{du}\left(\frac{dy}{dx}\right)\cdot\frac{du}{dx}$

SOLVED PROBLEMS

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$**,** given $x = \theta - \sin\theta$, $y = 1 - \cos\theta$

$$\frac{dx}{d\theta} = 1-\cos\theta, \quad \frac{dy}{d\theta} = \sin\theta, \quad \text{and} \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta}{1-\cos\theta}$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{\sin\theta}{1-\cos\theta}\right)\cdot\frac{d\theta}{dx} = \frac{\cos\theta-1}{(1-\cos\theta)^2}\cdot\frac{1}{1-\cos\theta} = -\frac{1}{(1-\cos\theta)^2}$$

12. FUNDAMENTAL INTEGRATION FORMULAS

IF *F(x) IS A* FUNCTION Whose derivative $F`(x)=f(x)$ on a certain interval of the *x*-axis, then *F(x)* is called an *anti-derivative* or *indefinite integral* of *f(x)*. The indefinite integral of a given function is not unique; for example, x^2, $x^2 + 5$, $x^2 - 4$ are indefinite integral of $f(x) = 2x$ since $\frac{d}{dx}(x^2) = \frac{d}{dx}(x^2+5) = \frac{d}{dx}(x^2-4) = 2x$. All indefinite integrals of $f(x) = 2x$ are then included in $x^2 + C$ where *C*, called the *constant of integration,* is an arbitrary constant.

The symbol $\int f(x)dx$ is used to indicate that the indefinite integral of *f(x)* is to be found. Thus we write $\int 2x\,dx = x^2 + C$

FUNDAMENTAL INTEGRATION FORMULAS. A number of the formulas below follow immediately from the standard differentiation formulas of earlier chapters while Formula 25, for example, may be checked by showing that

$$\frac{d}{du}\left\{\tfrac{1}{2}u\sqrt{a^2-u^2} + \tfrac{1}{2}a^2\arcsin\frac{u}{a} + C\right\} = \sqrt{a^2-u^2}$$

Absolute value signs appear in certain of the formulas. For example, we write

5. $$\int\frac{d}{du} = \ln|u| + C$$

instead of

5(a) $\int \frac{du}{u} = \ln u + C, \quad u > 0$ 5(b). $\int \frac{du}{u} = \ln(-u) + C, \quad u < 0$

and

10. $$\int \tan u \, du = \ln|\sec u| + C$$

instead of

10(a) $\int \tan u \, du = \ln \sec u + C$, all u such that $\sec u \geq 1$

10(b) $\int \tan u \, du = \ln(-\sec u) + C$, all u such that $\sec u \leq -1$

Fundamental Integration Formulas

1. $\int \frac{d}{dx}[f(x)]\,dx = f(x) + C$
2. $\int (u + v)\,dx = \int u\,dx + \int v\,dx$
3. $\int au\,dx = a\int u\,dx$, a any constant
4. $\int u^m\,du = \frac{u^{m+1}}{m+1} + C, \quad m \neq -1$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int a^u\,du = \frac{a^u}{\ln a} + C, \quad a > 0, a \neq 1$
7. $\int e^u\,du = e^u + C,$
8. $\int \sin u\,du = -\cos u + C$
9. $\int \cos u\,du = \sin u + C$
10. $\int \tan u\,du = \ln|\sec u| + C$
11. $\int \cot u\,du = \ln|\sin u| + C$
12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\int \csc u\,du = \ln|\csc u - \cot u| + C$
14. $\int \sec^2 u\,du = \tan u + C$
15. $\int \csc^2 u\,du = -\cot u + C$
16. $\int \sec u \tan u\,du = \sec u + C$

18. $\int \csc u \cot u\,du = -\csc u + C$

19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

20. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$

21. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arc\,sec} \frac{u}{a} + C$

22. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u-a}{u+a} \right| + C$

23. $\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \left| \frac{a+u}{a-u} \right| + C$

24. $\int \frac{du}{\sqrt{u^2 + a^2}} = \ln\left(u + \sqrt{u^2 + a^2} \right) + C$

25. $\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left| u + \sqrt{u^2 - a^2} \right| + C$

26. $\int \sqrt{a^2 - u^2}\, du = \frac{1}{2} u\sqrt{a^2 - u^2} + \frac{1}{2} a^2 \arcsin\frac{u}{a} + C$

27. $\int \sqrt{u^2 + a^2}\, du = \frac{1}{2} u\sqrt{u^2 + a^2} + \frac{1}{2} a^2 \ln\left(u + \sqrt{u^2 + a^2} \right) + C$

28. $\int \sqrt{u^2 - a^2}\, du = \frac{1}{2} u\sqrt{u^2 - a^2} - \frac{1}{2} a^2 \ln\left(u + \sqrt{u^2 - a^2} \right) + C$

13. INTEGRATION BY PARTS

INTEGRATION BY PARTS. When u and v are differentiable function of

$$d(uv) = u\,dv + v\,du$$

$$u\,dv = d(uv) - v(du)$$

(i) $$\int u\,dv = uv - \int v\,du$$

To use (i) in effecting a required integration, the given integral must be separated into two parts, one part being u and the other part, together with dx, being dv. (For this reason, integration by the use of (i) is called *integration by parts.*) Two general rules can be stated:

(a) the part selected as dv must be readily integrable

(b) $\int v\,du$ must not be more complex than $\int u\,dv$

Example 1: Find $\int x^3 e^{x^2} dx$

Take $u = x^2$ and $dv = e^{x^2} x\,dx$; then $du = 2x\,dx$ and $v = \frac{1}{2} e^{x^2}$. Now by the rule

$$\int x^2 e^{x^2} dx = \frac{1}{2} x^2 e^{x^2} - \int x\, e^{x^2} dx = \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C$$

Example 2: Find $\int \ln(x^2+2)dx$

Take u = ln (x^2 + 2) and $dv = dx;$ then $du = \frac{2x\,dx}{x^2+2}$ and $v = x$. By the rule.

$$\int \ln(x^2+2)dx = x\ln(x^2+2) - \int \frac{2x^2\,dx}{x^2+2} = x\ln(x^2+2) - \int\left(2 - \frac{4}{x^2+2}\right)dx$$

$$= x\ln\,(x^2+2) - 2x + 2\sqrt{2}\,arc\,\tan\,x/\sqrt{2} + C$$

REDUCTION FORMULAS. The labour involved in successive applications of integration by parts to evaluate an integral may be materially reduced by the use of *reduction formulas.* In general, a reduction formula yields a new integral of the same form as the original but with an exponent increased or reduced. A reduction formula succeeds if ultimately it produces an integral which can be evaluated. Among the reduction formulas are:

(A). $\int \frac{du}{(a^2 \pm u^2)^m} = \frac{1}{a^2}\left\{\frac{u}{(2m-2)(a^2 \pm u^2)^{m-1}} + \frac{2m-3}{2m-2}\int \frac{du}{(a^2 \pm u^2)^{m-1}}\right\}, \quad m \neq 1$

(B). $\int (a^2 \pm u^2)^m du = \frac{u(a^2 \pm u^2)^m}{2m+1} + \frac{2ma^2}{2m+1}\int (a^2 \pm u^2)^{m-1}du, \quad m \neq -1/2$

(C). $\int \frac{du}{(u^2 - a^2)^m} = -\frac{1}{a^2}\left\{\frac{u}{(2m-2)(u^2 - a^2)^{m-1}} + \frac{2m-3}{2m-2}\int \frac{du}{(u^2 - a^2)^{m-1}}\right\}, \quad m \neq 1$

(D). $\int (u^2 - a^2)^m du = \frac{u(u^2 - a^2)^m}{2m+1} - \frac{2ma^2}{2m+1}\int (u^2 - a^2)^{m-1}du, \quad m \neq -1/2$

(E). $\int u^m e^{au}\,du = \frac{1}{2}u^m e^{au} - \frac{m}{a}\int u^{m-1}e^{au}\,du$

(F). $\int \sin^m u\,du = -\frac{\sin^{m-1}u\cos u}{m} + \frac{m-1}{m}\int \sin^{m-2}u\,du$

(G). $\int \cos^m u\,du = \frac{\cos^{m-1}u\sin u}{m} + \frac{m-1}{m}\int \cos^{m-2}u\,du$

(H). $\int \sin^m u\cos^m u\,du = -\frac{\sin^{m+1}u\cos^{n-1}u}{m+n} + \frac{n-1}{m+n}\int \sin^m u\cos^{n-2}u\,du$

$$= -\frac{\sin^{m-1}u\cos^{n+1}u}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}u\cos^n u\,du, \quad m \neq -n$$

(I). $\int n^m \sin bu\,du = -\frac{u^m}{b}\cos bu + \frac{m}{b}\int u^{m-1}\cos bu\,du$

(J). $\int n^m \cos bu\,du = -\frac{u^m}{b}\sin bu - \frac{m}{b}\int u^{m-1}\sin bu\,du$

14. TRIGONOMETRIC INTEGRALS

THE FOLLOWING IDENTITIES are employed to find the trigonometric integrals of this chapter.

1. $\sin^2 x + \cos^2 x = 1$
2. $1 + \tan^2 x = \sec^2 x$
3. $1 + \cot^2 x = \csc^2 x$
4. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
5. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
6. $\sin x \cos x = \frac{1}{2}\sin 2x$
7. $\sin x \cos y = \frac{1}{2}[\sin(x - y) + \sin(x + y)]$
8. $\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$
9. $\cos x \cos y = \frac{1}{2}[\cos(x - y) + \cos(x + y)]$
10. $1 - \cos x = 2\sin^2 \frac{1}{2}x$
11. $1 + \cos x = 2\cos^2 \frac{1}{2}x$
12. $1 \pm \sin x = 1 \pm \cos(\frac{1}{2}\pi - x)$

SOLVED PROBLEMS

SINES AND COSINES

1. $\int \sin^2 x\, dx = \int \frac{1}{2}(1 - \cos 2x)\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

2. $\int \cos^2 3x\, dx = \int \frac{1}{2}(1 + \cos 6x)\, dx = \frac{1}{2}x - \frac{1}{12}\sin 6x + C$

3. $\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx = \int (1 - \cos^2 x)\sin x\, dx = -\cos x + \frac{1}{8}\cos^3 x + C$

4. $\int \cos^3 x\, dx = \int \cos^4 x \cos x\, dx = \int (1 - \sin^2 x)^2 \cos x\, dx$

$$= \int \cos x\, dx - 2\int \sin^2 x \cos x\, dx + \int \sin^4 x \cos x\, dx$$

$$= \sin x - \tfrac{2}{3}\sin^3 x + \tfrac{1}{5}\sin^3 x + C$$

5. $\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x \cos^2 x \cos x\, dx = \int \sin^2 x(1 - \sin^2 x)\cos x\, dx$

$$= \int \sin^2 x \cos x\, dx - \int \sin^4 x \cos x\, dx = \tfrac{1}{3}\sin^3 x - \tfrac{1}{5}\sin^5 x + C$$

15. TRIGONOMETRIC SUBSTITUTIONS

AN INTEGRAND, which contains one of the forms $\sqrt{a^2-b^2u^2}$, $\sqrt{a^2+b^2u^2}$, or $\sqrt{b^2u^2-a^2}$ but no other irrational factor, may be transformed into another involving trigonometric functions of a new variable as follows:

For	Use	To obtain
$\sqrt{a^2-b^2u^2}$	$u=\dfrac{a}{b}\sin z$	$a\sqrt{1-\sin^2 z}=a\cos z$
$\sqrt{a^2+b^2u^2}$	$u=\dfrac{a}{b}\tan z$	$a\sqrt{1+\tan^2 z}=a\sec z$
$\sqrt{b^2u^2-a^2}$	$u=\dfrac{a}{b}\sec z$	$a\sqrt{\sec^2 z-1}=a\tan z$

In each case, integration yields an expression in the variable *z*. The corresponding expression in the original variable may be obtained by the use of a right triangle as shown in the solved problems below.

SOLVED PROBLEMS

1. Find $\int \dfrac{dx}{x^2\sqrt{4+x^2}}$

Let $x = 2\tan z$; then $dx = 2\sec^2 z\,dz$ and $\sqrt{4+x^2}=2\sec z$

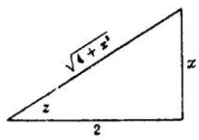


Fig. 28-1

$$\int \frac{dx}{x^2\sqrt{4+x^2}}=\int\frac{2\sec^2 z\,dz}{(4\tan^2 z)(2\sec z)}=\frac{1}{4}\int\frac{\sec z}{\tan^2 z}dz$$

$$=\frac{1}{4}\int \sin^{-2} z\cos z\,dz=-\frac{1}{4\sin z}+C=-\frac{\sqrt{4+x^2}}{4x}+C$$

2. Find $\int \dfrac{x^2}{\sqrt{x^2-4}}dx$

Let $x = 2\sec z$; then $dx = 2\sec z\tan z\,dz$ and $\sqrt{x^2-4}=2\tan z$

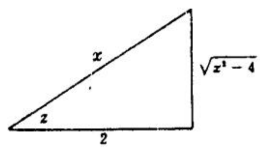


Fig. 28-2

$$\int \frac{x^2}{\sqrt{x^2-4}}dx=\int\frac{4\sec^2 z}{2\tan z}(2\sec z\tan z\,dz)=4\int\sec^3 z\,dz$$

$$=2\sec z\tan z+2\ln|\sec z+\tan z|+C$$

$$=\tfrac{1}{2}x\sqrt{x^2-4}+2\ln\left|x+\sqrt{x^2-4}\right|+C$$

3. Find $\int \frac{\sqrt{9-4x^2}}{x}dx$

Let $x = \frac{3}{2}\sin z$; then $dx = \frac{3}{2}\cos z\, dz$ and $\sqrt{9-4x^2} = 3\cos z$

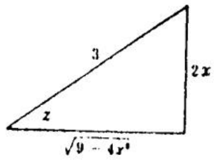


Fig. 28-3

$$\int \frac{\sqrt{9-4x^2}}{x}dx = \int \frac{3\cos z}{\frac{3}{2}\sin z}\left(\tfrac{3}{2}\cos z\, dz\right) = 3\int \frac{\cos^2 z}{\sin z}dz$$

$$= 3\int \frac{1-\sin^2 z}{\sin z}dz = 3\int \csc z\, dz - 3\int \sin z\, dz$$

$$= 3\ln\left|\csc z - \cot z\right| + 3\cos z + C$$

$$= 3\ln\left|\frac{3-\sqrt{9-4x^2}}{x}\right| + \sqrt{9-4x^2} + C$$

16. INTEGRATION BY PARTIAL FRACTIONS

A POLYNOMIAL IN *x* is a function of the form $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$, where the a's are constants, $a_0 \neq 0$, and n is a positive integer including zero.

If two polynomials of the same degree are equal for all values of the variable, the coefficients of the like powers of the variable in the two polynomials are equal.

Every polynomial with real coefficients can be expressed (at least, theoretically) as a product of real linear factors of the form $ax + b$ and real irreducible quadratic factors of the form $ax^2 + bx + c$.

A function $F(x) = \frac{f(x)}{g(x)}$**,** where $f(x)$ and $g(x)$ are polynomials, is called a *rational fraction.*

If the degree of $f(x)$ is less than the degree of $g(x)$, $F(x)$ is called *proper;* otherwise. $F(x)$ is called *improper.*

An improper rational fraction can be expressed as the sum of a polynomial and a proper rational fraction. Thus, $\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$.

Every proper rational fraction can be expressed (at least, theoretically) as a sum of simpler fractions *(partial fractions)* whose denominators are of the form $(ax + b)^n$ and $(ax^2 + bx + c)^n$, n being a positive integer. Four cases, depending upon the nature of the factors of the denominator, arise.

CASE I. DISTINCT LINEAR FACTORS

To each linear factor $ax + b$ occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\frac{A}{ax+b}$, where A is a constant to be determined.

CASE II. REPEATED LINEAR FACTORS

To each linear factor $ax + b$ occurring n times in the denominator of a proper rational fraction, there corresponds a sum of n partial fractions of the form

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\ldots+\frac{A_n}{(ax+b)^n}$$

where the A's are constants to be determined.

CASE III. DISTINCT QUADRATIC FACTORS

To each irreducible quadratic factor $ax^2 + bx + c$ occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined.

CASE IV. REPEATED QUADRATIC FACTORS

To each irreducible quadratic factor $ax^2 + bx + c$ occurring n times in the denominator of a proper rational fraction, there corresponds a sum of n partial fraction of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

where the A's and B's are constants to be determined.

SOLVED PROBLEMS

1. Find $\int\frac{dx}{x^2-4}$

   (a) Factor the denominator: $x^2 - 4 = (x-2)(x+2)$

   Write $\frac{1}{x^2-4}=\frac{A}{x-2}+\frac{B}{x+2}$ and clear of fraction to obtain

   (1) $1 = A(x + 2) + B(x - 2)$ or (2) $1 = (A + B)x + (2A - 2B)$

   (b) Determine the constants

   General method. Equate coefficients of like powers of x in (2) and solve simultaneously for the constants. Thus, $A + B = 0$ and $2A - 2B = 1$; $A=\frac{1}{4}$, and $B=-\frac{1}{4}$.

Short method. Substitute in (1) the values $x = 2$ and $x = -2$ to obtain $1 = 4A$ and $1 = -4B$; then $A = \frac{1}{4}$ and $B = -\frac{1}{4}$, as before. (Note that the values of x used are those for which the denominator of the partial fractions become 0).

(c) By either method: $\dfrac{1}{x^2-4} = \dfrac{\frac{1}{4}}{x-2} - \dfrac{\frac{1}{4}}{x+2}$ and

$$\int \frac{dx}{x^2-4} = \frac{1}{4}\int \frac{dx}{x-2} - \frac{1}{4}\int \frac{dx}{x+2} = \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$$

2. Find $\displaystyle\int \frac{(x+1)dx}{x^3+x^2-6x}$

(a) $x^3 + x^2 - 6x = x(x-2)(x+3)$. Then $\dfrac{x+1}{x^3+x^2-6x} = \dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{x+3}$ and

$(1)\ x + 1 = A(x-2)(x+3) + Bx(x+3) + Cx(x-2)$ or

$(2)\ x + 1 = (A + B + C)x^2 + (A + 3B - 2C)x - 6A$

(b) *General method.* Solve simultaneously the system of equation

$A + B + C = 0,\quad A + 3B - 2C = 1,$ and $-6A = 1$

To obtain $A = -1/6$, $B = 3/10$, and $C = -2/15$.

Short method. Substitute in (1) the values $x = 0$, $x = 2$, and $x = -3$ to obtain $1 = -6A$ or $A = -1/6$, $3 = 10B$ or $B = 3/10$, and $-2 = 15C$ or $C = -2/15$

(c) $$\int \frac{(x+1)dx}{x^3+x^2-6x} = -\frac{1}{6}\int \frac{dx}{x} + \frac{3}{10}\int \frac{dx}{x-2} - \frac{2}{15}\int \frac{dx}{x+3}$$

$$= -\frac{1}{6}\ln|x| + \frac{3}{10}\ln|x-2| - \frac{2}{15}\ln|x+3| + C = \ln\frac{|x-2|^{3/10}}{|x|^{1/6}|x+3|^{2/15}} + C$$

3. Find $\displaystyle\int \frac{(3x+5)dx}{x^3-x^2-x+1}$

$x^3 - x^2 - x + 1 = (x+1)(x-1)^2$. Then $\dfrac{3x+5}{x^3-x^2-x+1} = \dfrac{A}{x+1} + \dfrac{B}{x-1} - \dfrac{C}{(x-1)^2}$ and

$$3x + 5 = A(x-1)^2 + B(x+1)(x-1) + C(x+1).$$

For $x = -1$, $2 = 4A$ and $A = ½$. For $x = 1$, $8 = 2C$ and $C = 4$. To determine the remaining constant, use any other value of x, say $x = 0$; for $x = 0$,
$5 = A - B + C$ and $B = -½$. Thus

$$\int \frac{(3x+5)dx}{x^3-x^2-x+1} = \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{x-1} + 4\int \frac{dx}{(x-1)^2}$$

$$= \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| - \frac{4}{x-1} + C$$

$$= -\frac{4}{x-1} + \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C$$