CHAPTER 4

LINE PERPENDICULAR TO PLANE

LEARNING OBJECTIVES

- mention the definition of line perpendicular to plane
- prove that a line perpendicular to a plane
- understand theorems related to line perpendicular to plane
- determine the projection of a point to a plane

OVERVIEW

Line perpendicular to a plane is a special case of line intersect plane.

Definition. If a straight line drawn to a plane is perpendicular to every straight line that passes through its foot and lies in the plane, it is said to be **perpendicular to the plane**.

When a line is perpendicular to a plane, the plane is also said to be **perpendicular to the line**.

Theorem. If a line is perpendicular to each of two other lines at their point of intersection, it is perpendicular to the plane of the two lines.

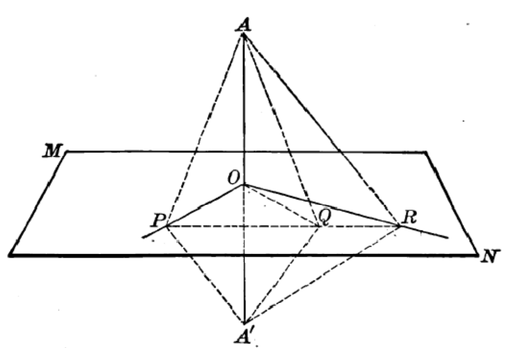

Figure 1.

Given the line AO perpendicular to the lines OP and OR at O. It will be proved that AO perpendicular to the plane MN of these lines.

Through O draw in MN any other line OQ, and draw PR cutting OP, OQ, OR, at P, Q, and R. Produce AO to A', making OA' equal to OA, and join A and A' to each of the points P, Q, and R. Then OP and OR are each perpendicular to AA' at its mid-point. So,

AP=A'P and AR=A'R

⇒Triangle APR is congruent to triangle A'PR

⇒∠RPA=∠RPA'

That is ∠QPA=∠QPA'.

Triangle PQA is congruent to triangle PQA'.

AQ=A'Q

OQ is perpendicular to AA' at O.

AO is perpendicular to any and hence to every line in MN through O.

AO is perpendicular to the plane MN.

Theorem. Let n be a line perpendicular to two intersecting lines k and l in the plane α. Then n is perpendicular to every line in α, hence n ⊥ α.

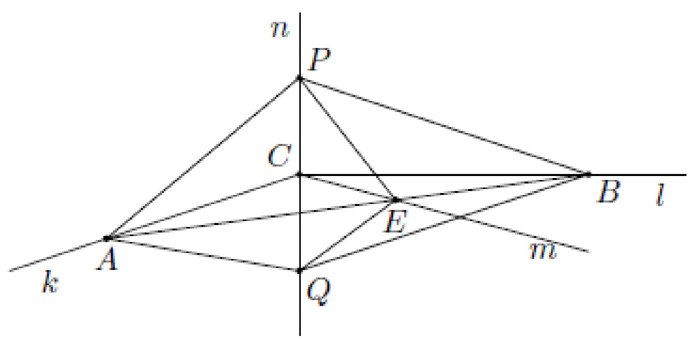

Figure 2. If n ⊥ k and n ⊥ l then n ⊥ α

Theorem. For every line l and every point P there is a unique plane α through P such that l ⊥ α.

Theorem. Two planes α and β that are perpendicular to the same line n are parallel.

We use this theorem to define the dihedral angle between two intersecting Planes that will be discussed on the next chapter.

GROUP ASSIGNMENT

On cube ABCD.EFGH Prove that :

1. HD is perpendicular to ABCD
2. HD is perpendicular to AC
3. AD is perpendicular to GH
4. AF is perpendicular to BCHE
5. AF is perpendicular to HC

Theorem. All the perpendiculars that can be drawn to a given point lie in a plane which is perpendicular to the given line at the given point.

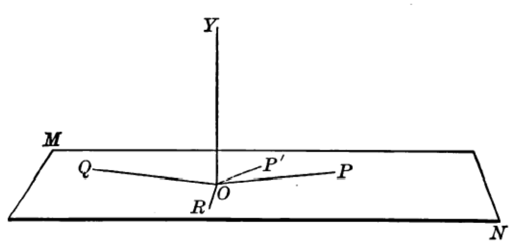

Figure 3.

Proof. Given the plane MN perpendicular to the line OY at O. It will be proved that OP, any line perpendicular to OY at O lies in MN. Let the plane containing OY and OP intersect the plane MN in the line OP', then OY is perpendicular to OP'. in the plane POY only one perpendicular can be drawn to OY at O. therefore OP and OP' coincide, and OP lies in MN. Hence every perpendicular to OY at O, OQ, OR, lies in MN.

Corollary. Through a given point in a given line one plane, and only one, can be passed perpendicular to the line.

Through a given external point one plane, and only one, can be passed perpendicular to a given line.

A line that meets a plane but is not perpendicular to it is said to be **oblique** to the plane.

Theorem. Through a given point in a plane there can be drawn one line perpendicular to the plane, and only one.

Through a given external point there can be drawn one line perpendicular to a given plane, and only one.

Corollary. The perpendicular is the shortest line from a point to a plane.

The length of this perpendicular is called the **distance** from the point to the plane.

Theorem. Two planes α and β that are perpendicular to the same line n are parallel.

We use this theorem to define the dihedral angle between two intersecting planes that will be discussed on the next chapter.

Theorem. Oblique lines drawn from a point to a plane, meeting the plane at equal distances from the foot of the perpendicular, are equal; and of two oblique lines, meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the greater.

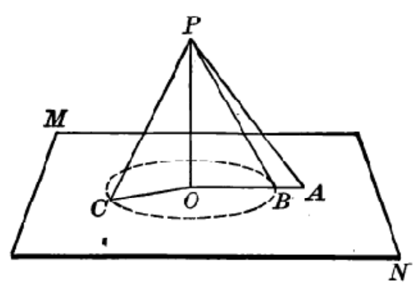

Figure 4.

On the figure, PB=PC and PA>PC.

Corollary. Equal oblique lines drawn from a point to a plane meet the plane at equal distances from the foot of the perpendicular; and of two unequal oblique lines the greater meets the plane at the greater distance from the foot of the perpendicular.

Corollary. The locus of a point equidistant from all points on a circle is a line through the center, perpendicular to the plane of the circle.

Corollary. The locus of a point equidistant from the vertices of a triangle is a line through the center of the circumscribed circle, perpendicular to the plane of the triangle.

Theorem. Two lines perpendicular to the same plane are parallel.

Corollary. If one of two parallel is perpendicular to a plane, the other is also perpendicular to a plane.

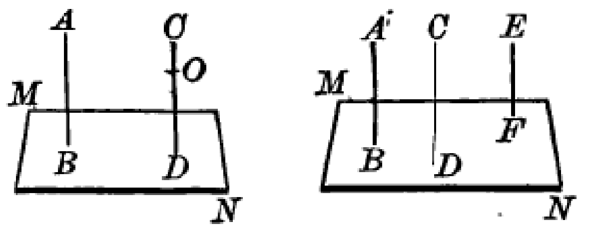

Figure 5.

If two lines are parallel to a third line, they are parallel to each other.

If a line and plane cannot meet, they are said to be **parallel**.

Theorem. If two lines are parallel, every plane containing one of the lines, and only one, is parallel to the other line.

Corollary. Through either of two lines not in the same plane one plane, and only one, can be passed parallel to the other.

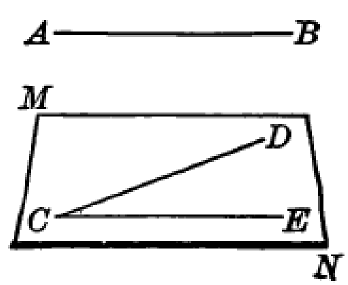

Figure 6.

Corollary. Through a given point one plane, and only one, can be passed parallel to any two given lines in space.

Two planes which cannot meet are said to be **parallel**.

Two planes perpendicular to the same line are **parallel**.

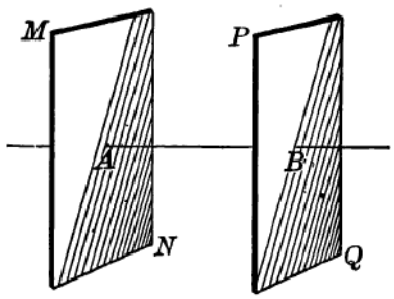

Figure 7.

Theorem. The intersections of two parallel planes by a third plane are parallel lines.

Corollary. Parallel lines included between parallel planes are equal.

Corollary. Two parallel planes are everywhere equidistant from each other.

Theorem. A line perpendicular to one of two parallel planes is perpendicular to the other also.

Corollary

Through a given point one plane, and only one, can be passed parallel to a given plane.

The locus of a point equidistant from two parallel planes is a plane perpendicular to a line which is perpendicular to the planes and which bisects the segment cut off by them.

The locus of a point equidistant from two parallel lines is a plane perpendicular to a line which is perpendicular to the given lines and which bisects the segment cut off by them.

Theorem

If two intersecting lines are each parallel to a plane, the plane of these lines is parallel to that plane.

If two lines are cut by three parallel planes, their corresponding segments are proportional.

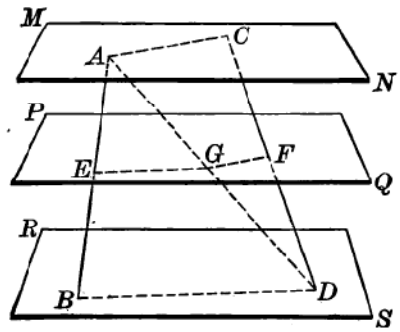

Figure 8

ORTHOGONAL PROJECTION

Theorem

Through a given oblique or parallel to a plane one and only one plane can be passed perpendicular to the given plane.

The orthogonal projection of a point on a plane is the foot of a perpendicular let fall on the plane from the point.

The orthogonal projection on an oblique or a parallel line is the intersection of the plane with a perpendicular plane passed through the line. Evidently this projection is a stright line.

The othogonal projection of a line-segment is the locus of the projections of its points.

Theorem. The acute angle formed by a line and its orthogonal projection upon a plane is the least angle which the line makes with any line in the plane.

The angle between a line and a plane is the angle between the line (produced if necessary) and its orthogonal projection on the plane.

RECOMMENDED READING

Beman WW and Smith, David Eugene. New Plane and Solid Geometry. Ginn and Company: Boston