The General Second-Order System

- Two important quantities that describes the response of second order systems:
  - *Natural frequency*, ω_n: The frequency of oscillation of the system without damping.
  - *Damping ratio*, ζ: Parameter that describes the damped oscillations of the 2nd order response. Bigger, means more 'damped' response, i.e. less oscillations.

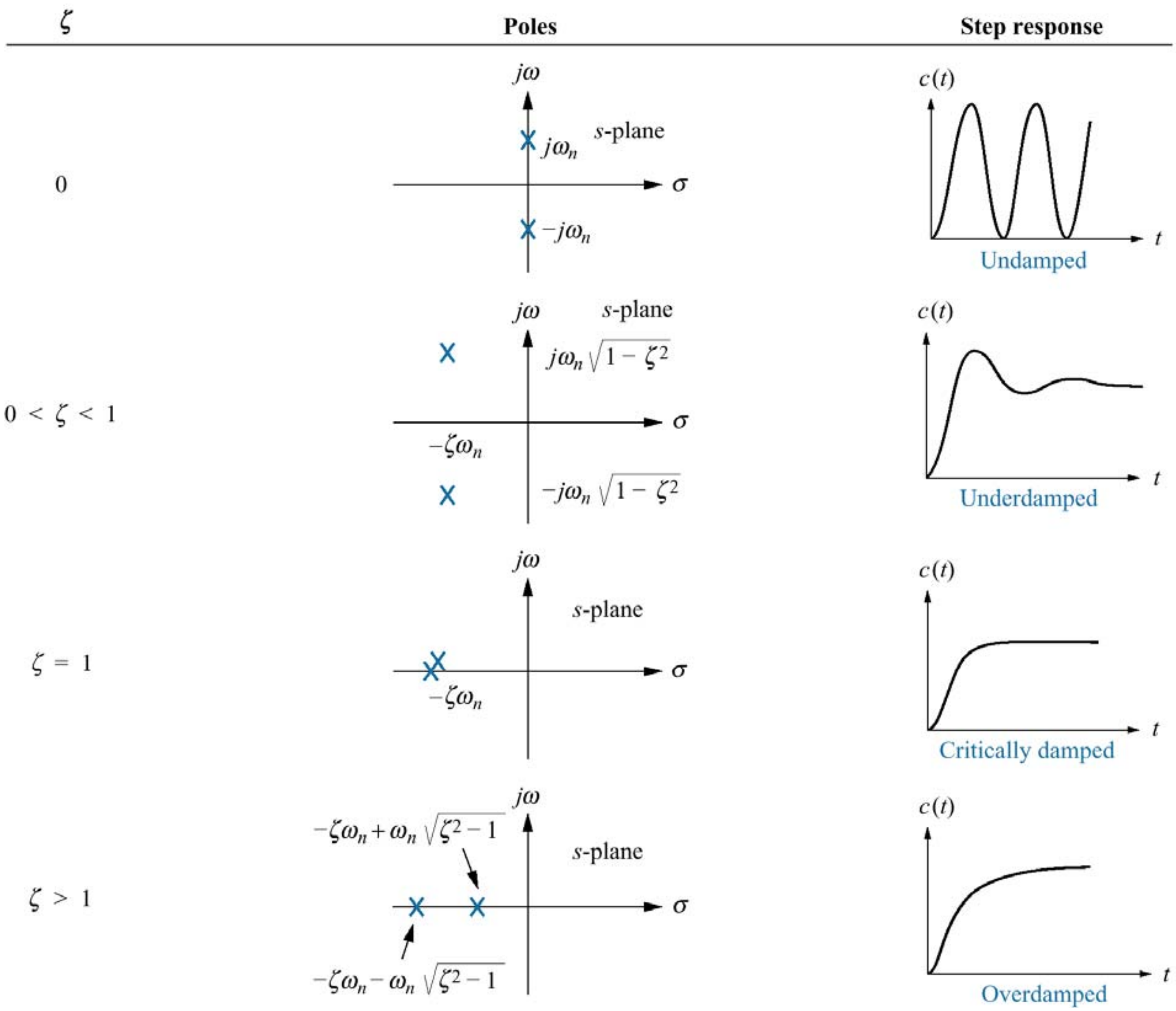
$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/second)}}$$

$$= \frac{1}{2\pi}\frac{\text{Natural period (seconds)}}{\text{Exponential time sonstant}}$$

- Define general 2nd order response in terms of ω_n and ζ as:

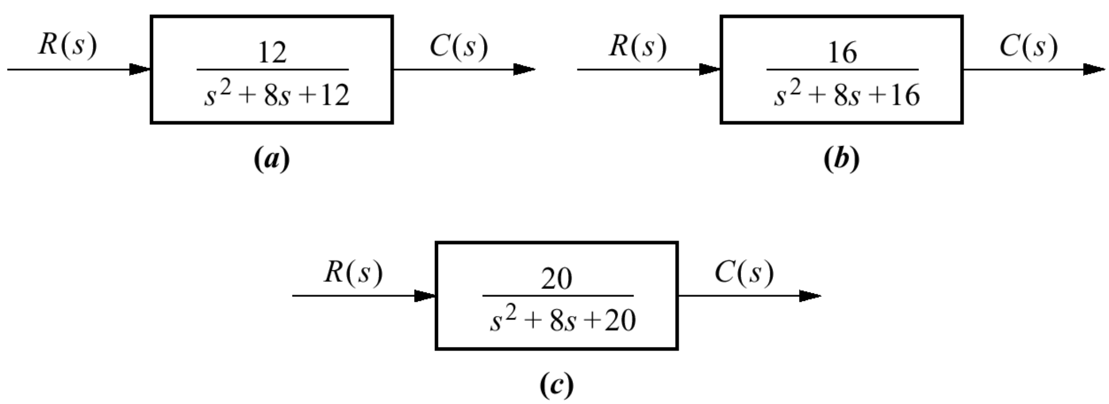
- Hence the pole is given as:

- Four time responses based on ζ :



<u>Example:</u> Given the transfer function $G(s) = \dfrac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$, find ζ and ω_n.

<u>Example:</u> Find the value of ζ, and sketch the kind of response expected.



Underdamped Second-Order Systems

- A common model for physical problems.
- A detailed description of the underdamped response is necessary for both analysis and design.

$$C(s) = \frac{\omega_n^{\ 2}}{s(s^2 + 2\zeta\omega_n s + \omega_n^{\ 2})}$$

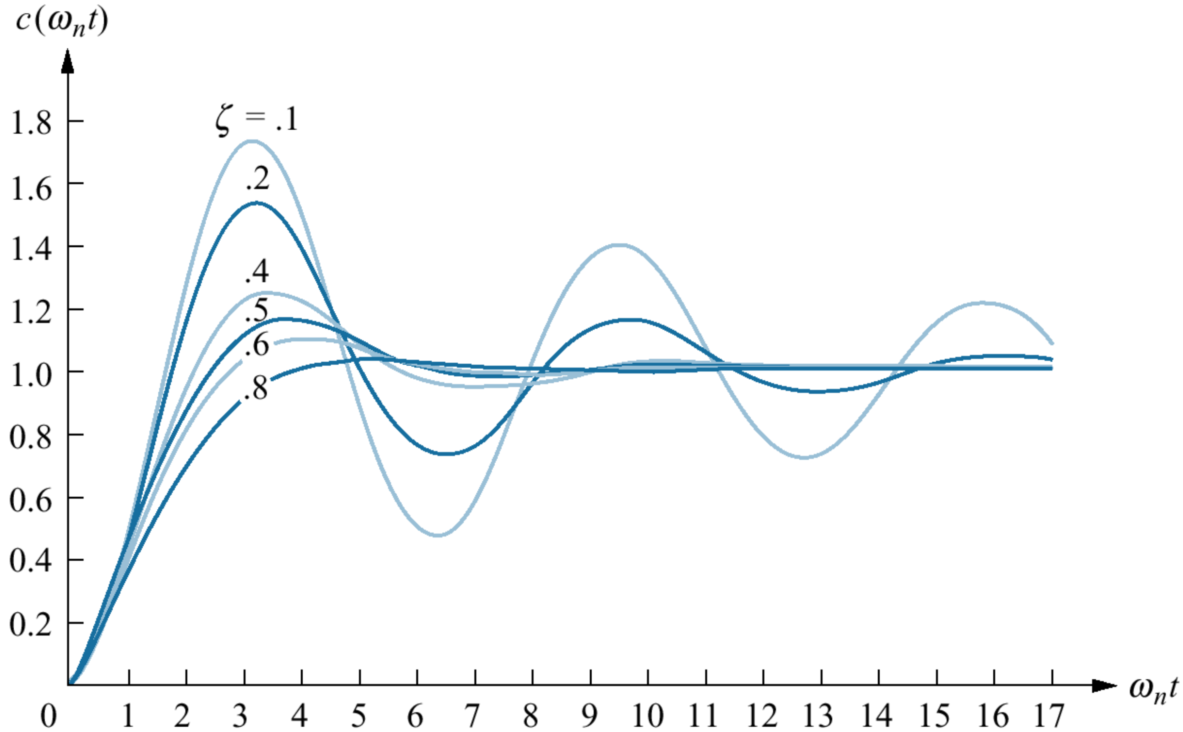
$$= \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}, \quad \zeta < 1$$

$$= \frac{1}{s} - \frac{(s+\zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\,\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^{\ 2}\sqrt{1-\zeta^2}}$$

- Taking the inverse Laplace transform,

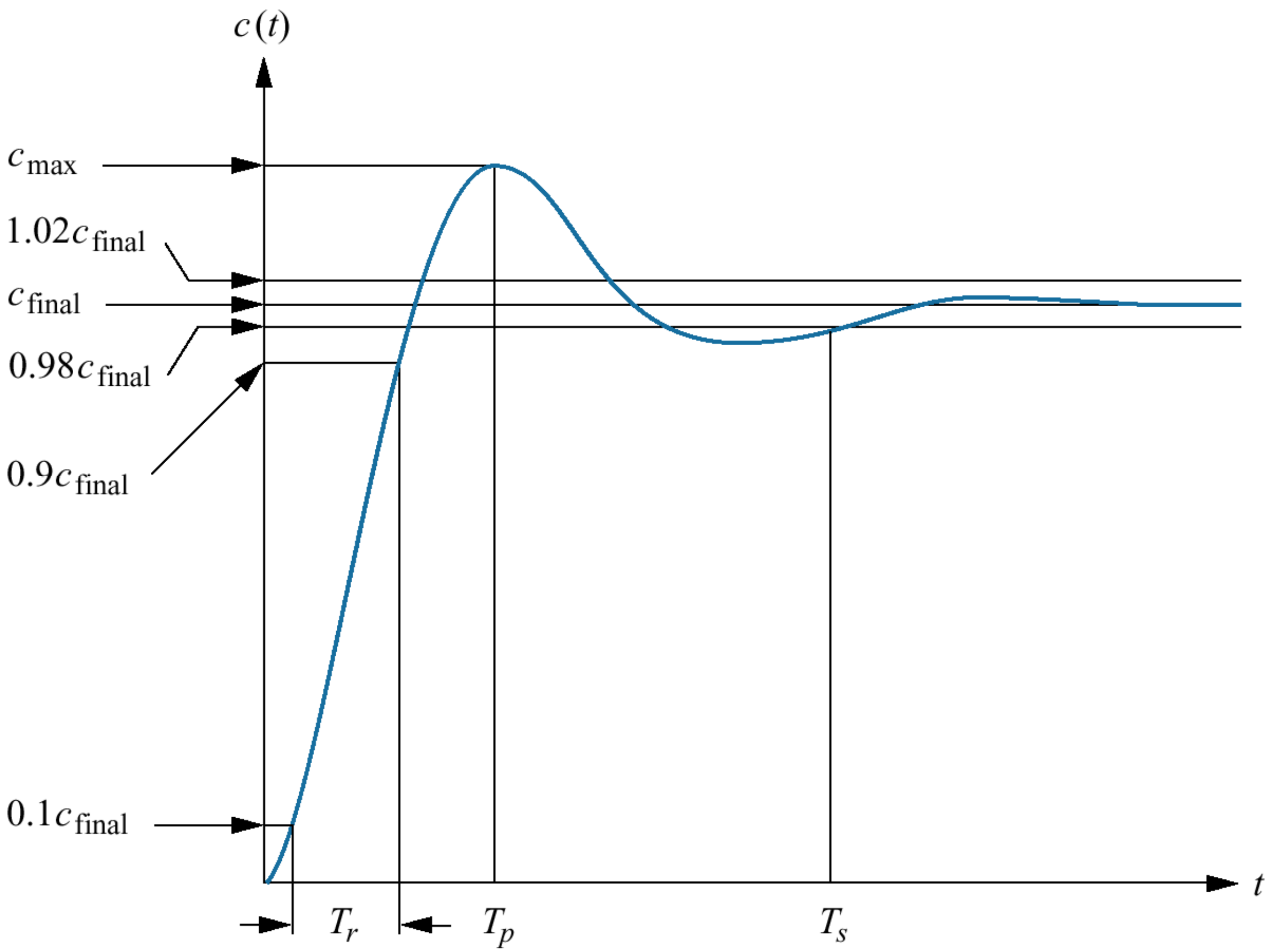
$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t\right)$$

- plot of *c(t)*



- Less ζ implies more oscillation.

Second order underdamped response specifications

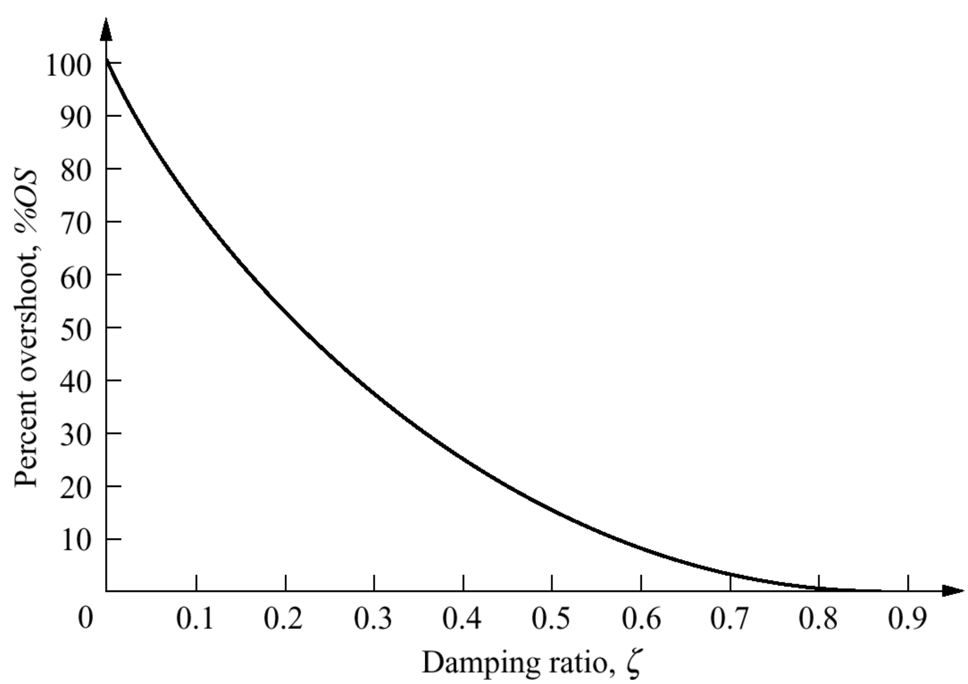


- **Peak time, T_p**: The time required to reach the first, or maximum, peak.

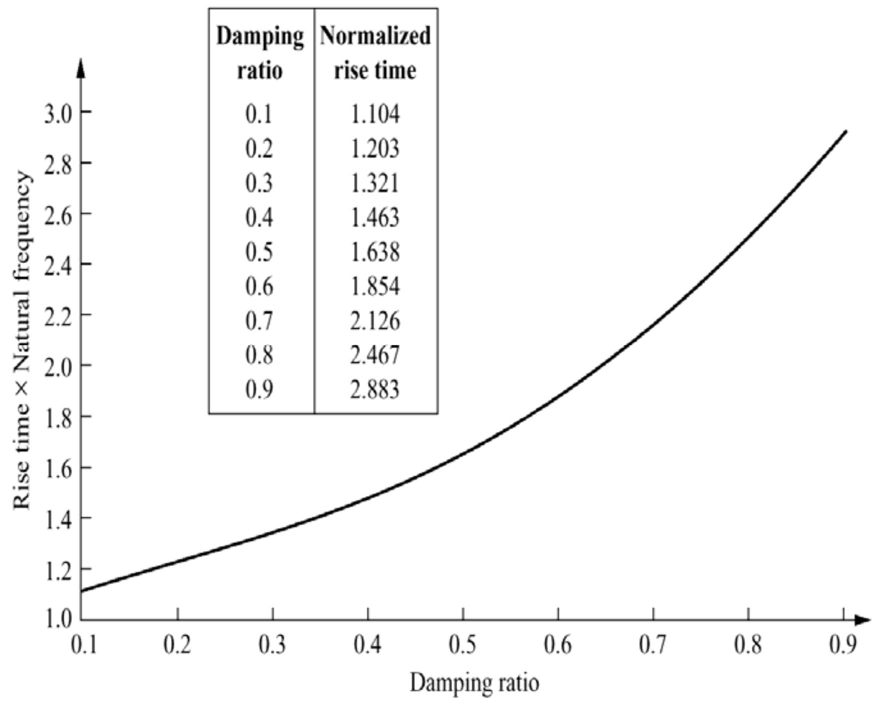
- **Percent overshoot, %OS:** The amount that the response overshoots the final value at the peak time, expressed as a percentage of the steady-state value.

- We can also find the inverse of the equation allowing us to find ζ given %OS.

- Relationship between ζ and %OS can be used:



- **Rise time, T_r:** The time required for the response to go from 0.1 to 0.9 of the final value.
- Relationship between $T_r\omega_n$ and ζ can be used.



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

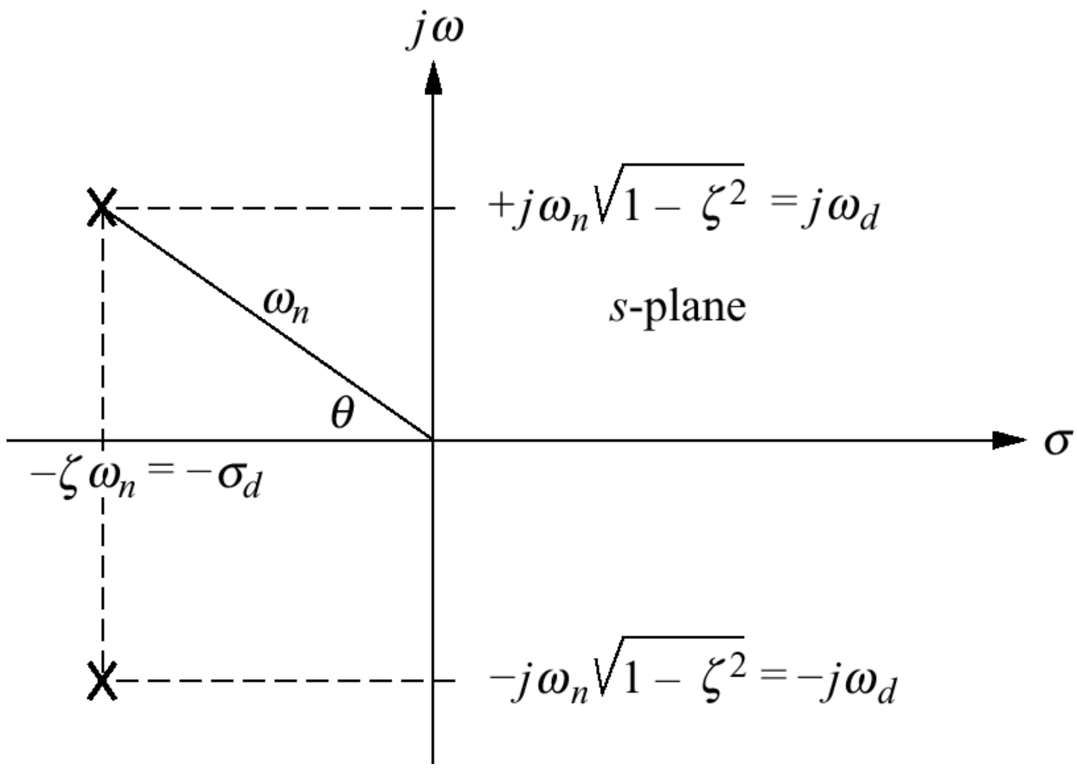
- **Settling time, T_S:** The time required for the response to reach and stays within $\pm 2\%$ of the steady-state value/final value.

Example:

Given a transfer function, $G(s) = \dfrac{100}{s^2 + 15s + 100}$, find T_p, $\%OS$, T_S, and T_r.

Relation between T_p, %OS, T_S, and T_r to the system poles.

- Consider the pole plot for an underdamped second-order system:



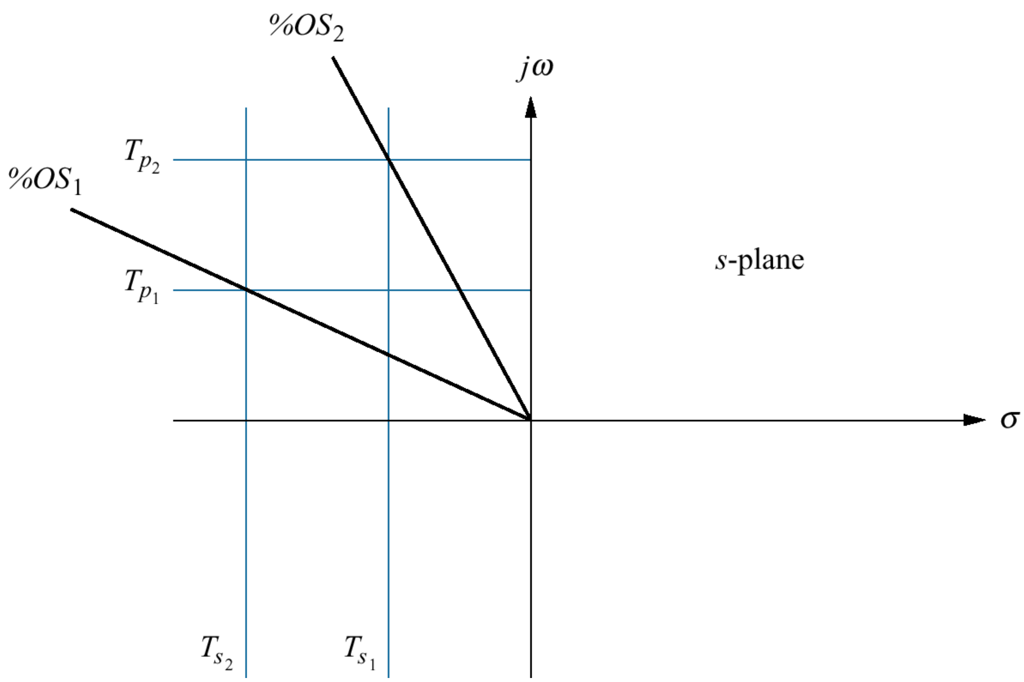
- Notice that the distance from the origin to the pole equals ω_n.

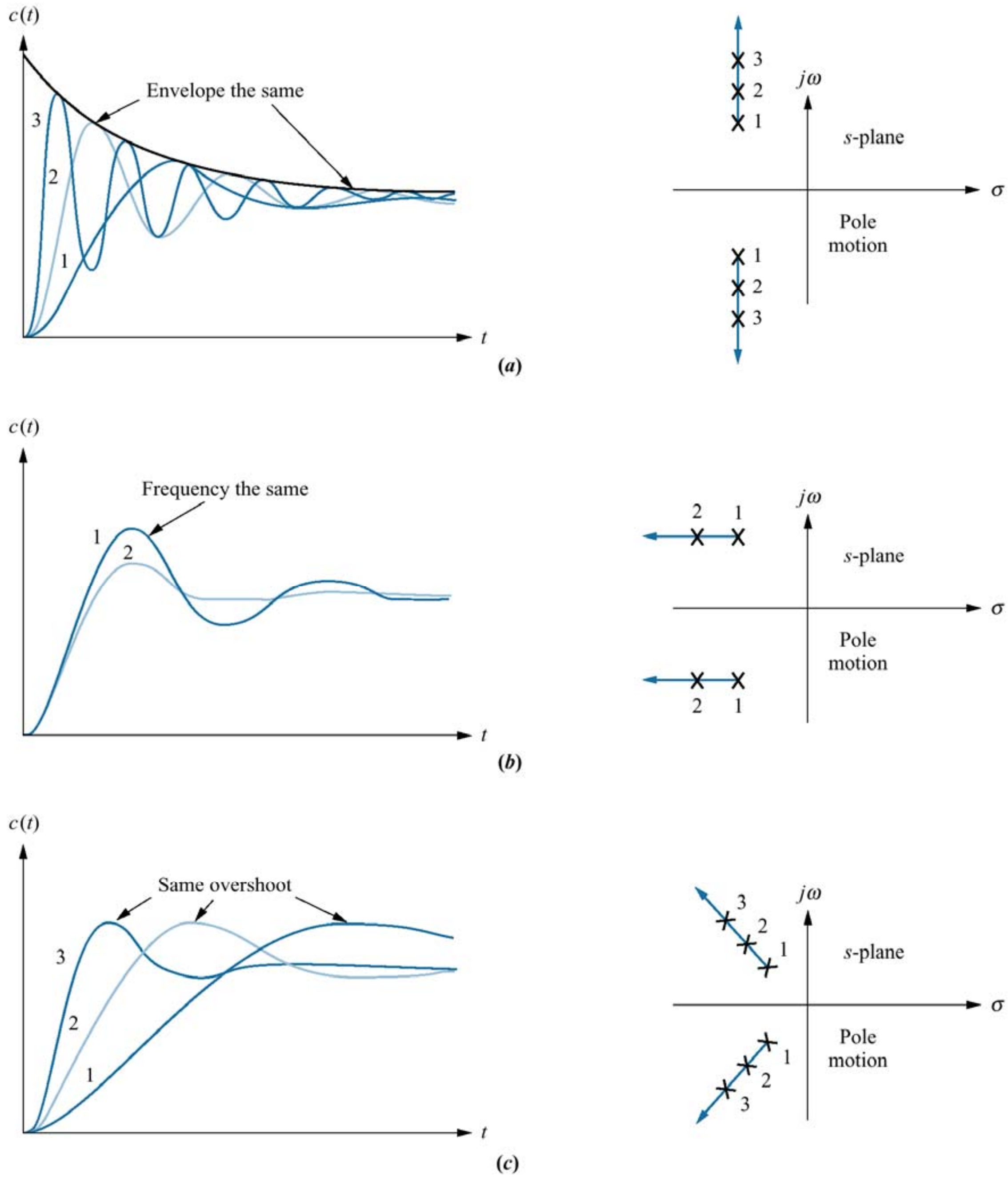
- $\cos\theta = \dfrac{-\zeta\omega_n}{\omega_n} = \zeta$.

- Before we already defined:

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}, T_S = \frac{4}{\zeta\omega_n}$$

- $\omega_d = \omega_n \sqrt{1-\zeta^2}$ is the imaginary part of the pole, called the *damped frequency of oscillation*.
- T_P is inversely proportional to the imaginary part of the pole. $\Rightarrow$Horizontal lines are lines of constant peak time.
- T_S is inversely proportional to the magnitude of the real part of the pole. $\Rightarrow$Vertical lines are lines of constant settling time.
- Radial lines are lines of constant ζ. $\Rightarrow$Radial lines are lines of constant %OS.

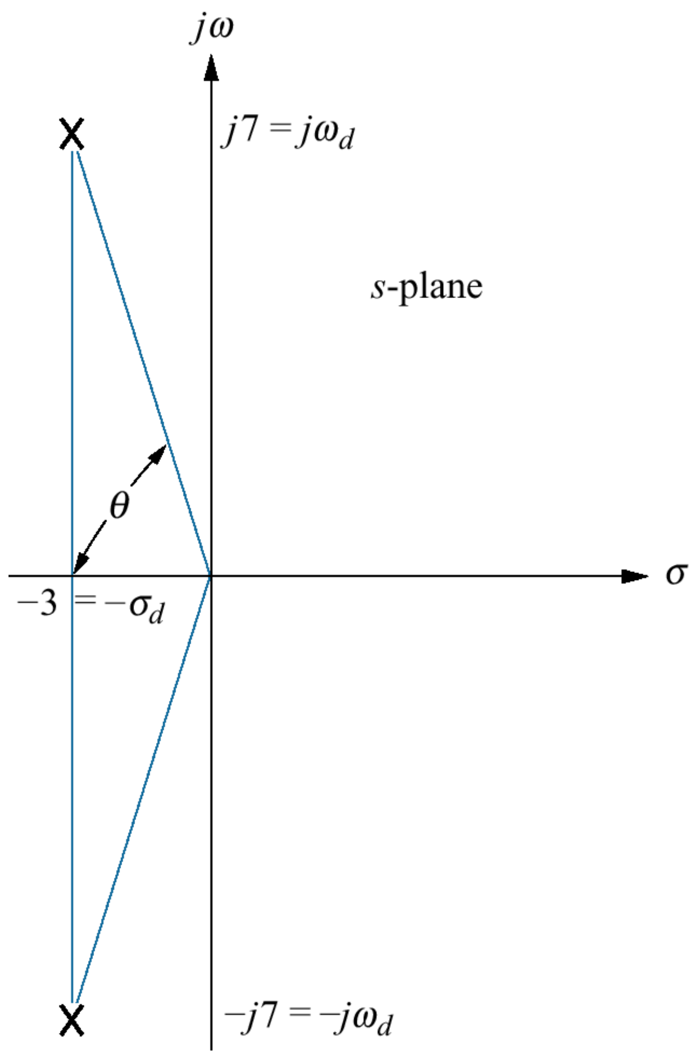




- Step responses of second order underdamped systems as pole moves: (a) with constant real part, (b) with constant imaginary part, (c) with constant damping ratio.

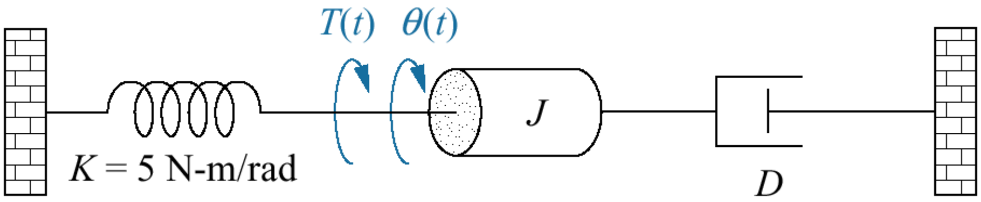
Example:

Given the pole plot shown, find ζ, ω_n, T_p, %OS, and T_S.



Example:

For the system shown, find J and D to yield 20% overshoot and a settling time of 2 seconds for a step input of torque *T(t)*.



Example:

For the unit step response shown, find the transfer function of the system.