LECTURE NOTES

METODE PENGUKURAN FISIKA (FIC-213)

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Physical Measurement Method
(FIC-213)
2 units

Denny Darmawan

Textbook:
An Introduction To Error Analysis
By **John R. Taylor**

The study and evaluation of uncertainties
in physical measurements
(a.k.a *Error Analysis*)

20% Assignments
+
30% Mid-term Exam
+
50% Final Exam

Physical Measurement Method

[Chapter 1]

[1.1 – 1.2] inevitability of uncertainty

No measurement can be completely free of uncertainties

Error ≠ mistake

Error = inevitable uncertainty attending all measurement

Errors cannot be eliminated by being very careful in conducting measurement

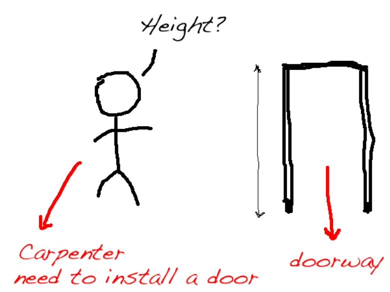
The best we can do is:

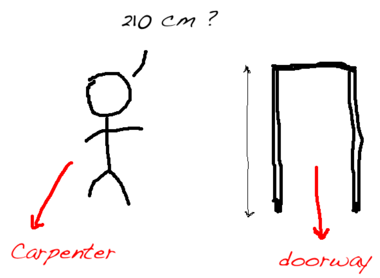
to ensure that errors are as small as possible

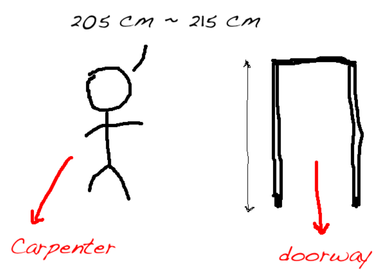
+

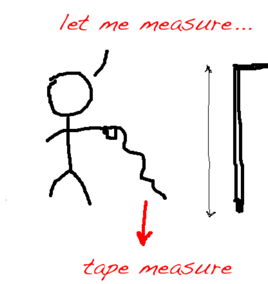
to have reliable estimate of how large they are

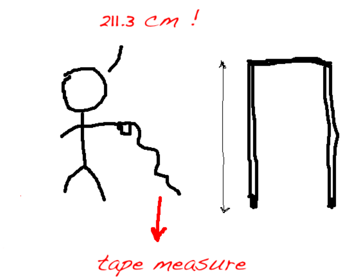
Illustration:

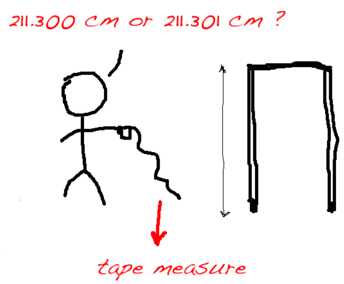


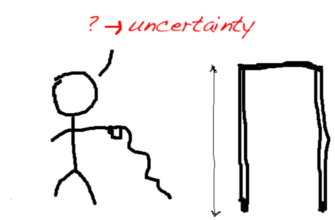








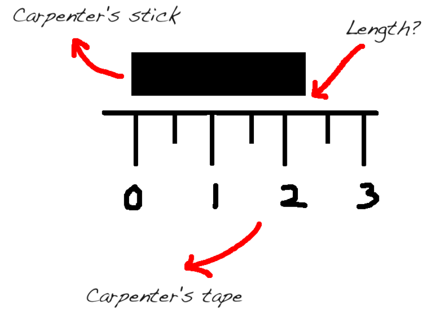


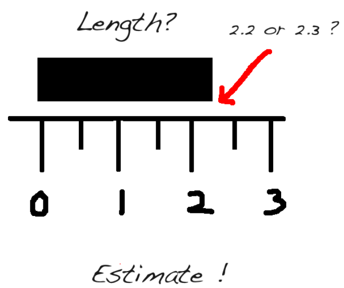


Some sources of uncertainty are intrinsic to the process of measurement

→ cannot be removed entirely

Illustration: back to our carpenter





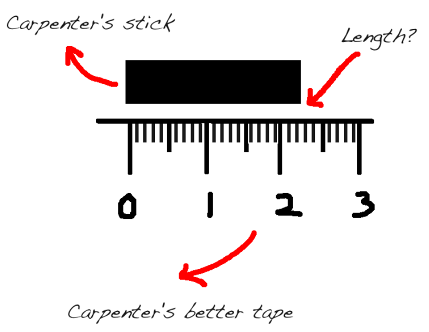
Solution?

Buy a better tape
with closer and finer marking!

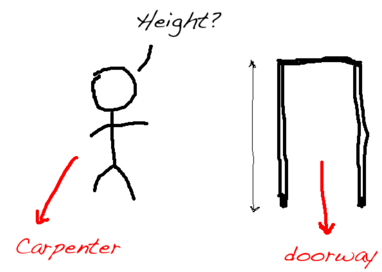
Further solution:

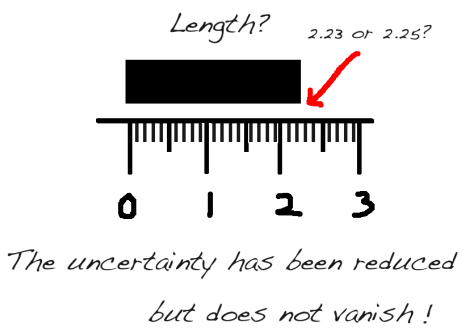
buy a laser interferometer!

Often the uncertainties are important
but can be allowed for instinctively and
without explicit consideration



The precision of an interferometer
is limited to distances of the order of
the wavelength of light
(0.5 nanometers)





The carpenter can reduce his uncertainty
but cannot eliminate it entirely

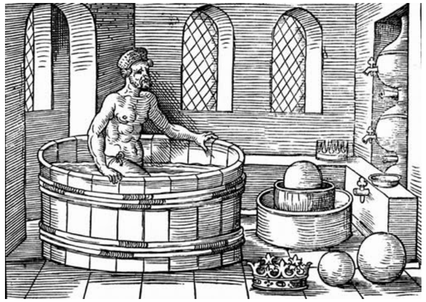
Our carpenter must know the height of the doorway with an uncertainty that is less than 1 mm or so..

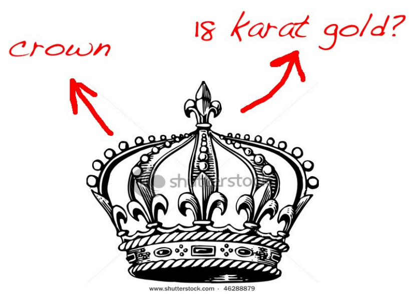
As long as the uncertainty is this small, the door will be a perfect fit

(and his concern with error analysis is at an end..)

Another example...

Archimedes' Experiment





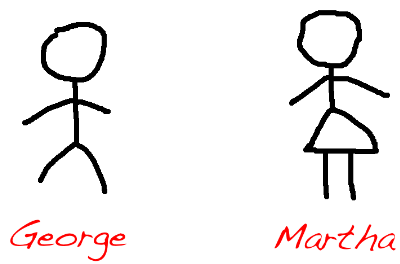
Facts:

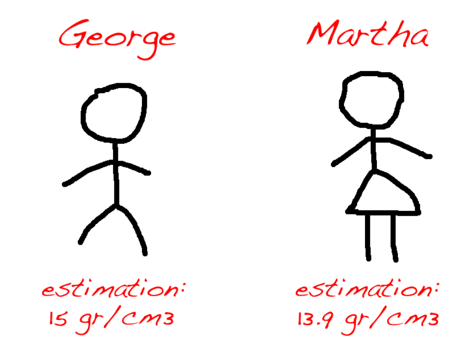
Density of 18-karat gold: 15.5 gr/cm^3

Density of alloy: 13.8 gr/cm^3

Measure the density of the crown, and it tells whether the crown is really gold or just a cheaper alloy...

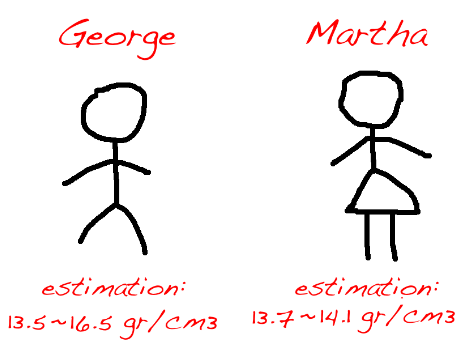
Our Experimenters:





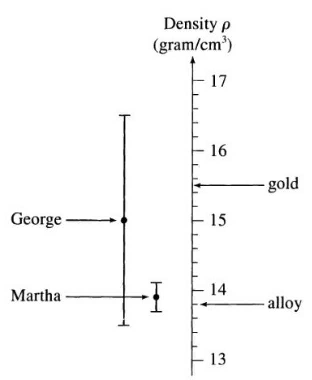
The uncertainty in George's measurement is so large

Martha's measurement indicate clearly that the crown is not genuine



The densities of 18-karat gold and of the alloy both lie within George's range, from 13.5 to 16.5 gr/cm^3

The density of alloy (13.8) lies inside Martha's estimated range of 13.7 to 14.1



No conclusion can be drawn from George's measurement!

The density of 18-karat (15.5) lies far outside Martha's estimated range of 13.7 to 14.1

Uncertainties have to be reasonably small,
But extreme precision is often unnecessary

Because we depend on Martha's result,
she must give us sufficient reason
to believe her claim

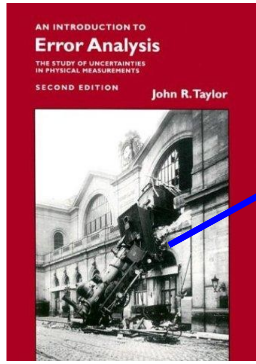
→ she must justify her stated range of values for ρ

In making the estimation, we have to give
a brief explanation of how the uncertainty
was estimated

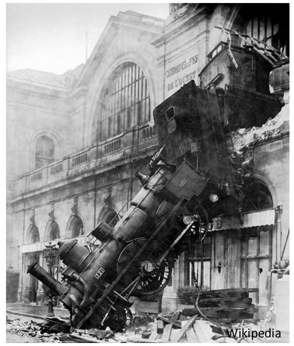
[1.3 – 1.4] importance of knowing
the uncertainties

Engineers concerned with the safety
of airplanes, trains or cars
must understand the uncertainties
in drivers' reaction time,
in braking distances,
and in a host of other variable

Failure to carry out error analysis
can lead to accidents...



1895 Paris train disaster



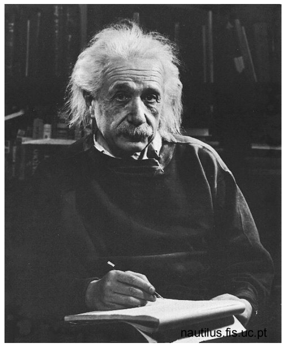
Train derailment at Gare Montparnasse, Paris, 1895.
The accident was caused by a faulty brake and late driver's reaction time

In basic science, any new theory proposed
must be tested against older theory
by means of experiments

The experimental results must be consistent with the prediction of one theory and inconsistent with the others

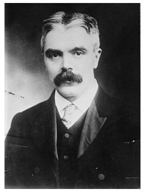
Example:

The measurement of the bending of light as it passes near the sun

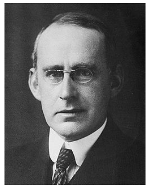


Proposed by Einstein in 1916 using his general theory of relativity

measured experimentally in 1919 by:

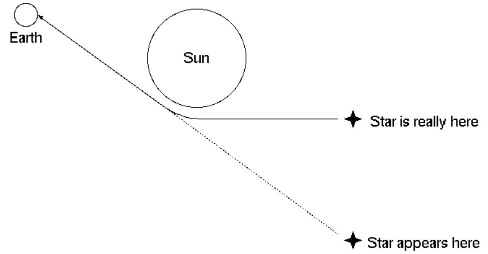


Frank Dyson
(1868 – 1939)



Arthur Eddington
(1882 – 1944)

C. Davidson
(? - ?)



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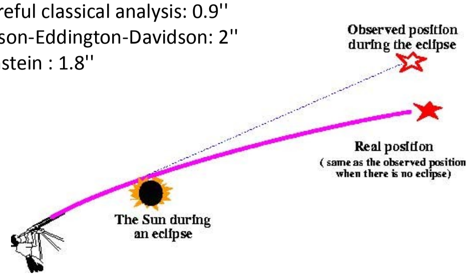
The angle of bending:

Simplest classical analysis: 0''
Careful classical analysis: 0.9''
Dyson-Eddington-Davidson: 2''
Einstein : 1.8''



sunearthday.nasa.gov

Measurement result
from Dyson-Eddington-Davidson:

Best estimate: 2''
Estimation: between 1.7'' and 2.3''

→ consistent with general relativity theory and inconsistent with either of the older predictions

[1.5] estimating uncertainties
when reading scales

Estimate!

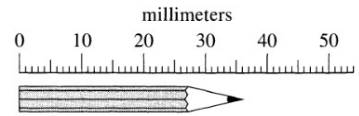


Figure 1.2. Measuring a length with a ruler.

- place the end of the pencil opposite the zero of the ruler
- decide where the tip comes to on the ruler's scale

The markings are 1 mm apart

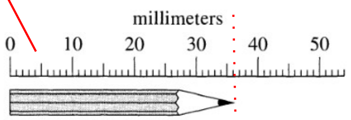


Figure 1.2. Measuring a length with a ruler.

Closer to 36 mm than it is to 35 or 37 mm

In the same way...

$x = 1.27$
(without any stated uncertainty)

means

x lies between 1.265 and 1.275

[1.6] estimating uncertainties in repeatable measurement

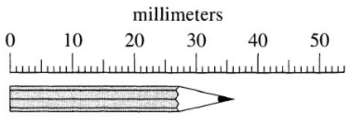


Figure 1.2. Measuring a length with a ruler.

Best estimate of length = 36 mm
probable range: 35.5 to 36.5 mm

Estimate!

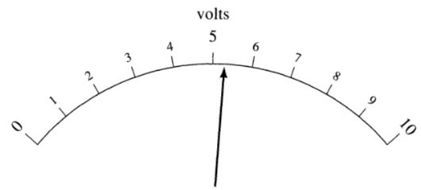


Figure 1.3. A reading on a voltmeter.

In measuring a time interval using a stopwatch,

the main source of uncertainties is not the difficulty in reading the dial

Convention introduced by scientists:

$l = 36$ mm

means

$35.5 \text{ mm} \leq l \leq 36.5 \text{ mm}$

Best estimate: 5.3 volts
probable range: 5.2 to 5.4 volts

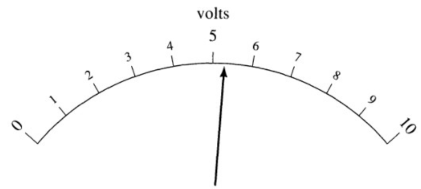
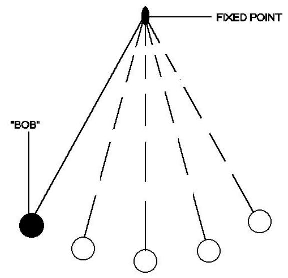


Figure 1.3. A reading on a voltmeter.

→ the process of estimating positions between the scale markings is called *interpolation*

but our own
unknown
reaction time
in starting
and stoping
the watch



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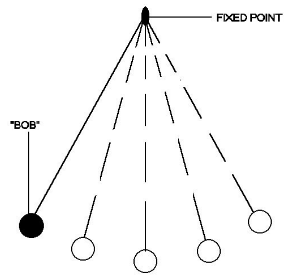
Example:

Time the period of a pendulum once

Suppose we get 2.3 seconds

But we can't say much about the experimental uncertainties!



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Repeat the measurement

Suppose we get 2.4 seconds

We can say that the uncertainty is probably of the order of 0.1 s

If a sequence of four timing gives this results (in seconds):

2.3, 2.4, 2.5, 2.4

Then the best estimate of the period is the average value: 2.4 s

Probable range: 2.3 to 2.5 s

This estimation is just a simple and realistic conclusion to draw from measurements

Deeper analysis on repeated measurement results will be discussed in Chapter 4 – 5

Repeated measurement cannot always be relied on to reveal the uncertainties

We must be sure that the quantity measured is really *the same* quantity each time

Errors which affect all measurements in the same way are called *systematic errors*

Example:

Suppose the clock used for the timing was running consistently 5% fast

then all timing will be 5% too long

and no amount of repeating will reveal this deficiency

Physical Measurement Method

[Chapter 2]

[2.1] Best Estimate ± Uncertainty

From Previous Discussion...

Best estimate of time = 2.4 s

Probable range: 2.3 to 2.5 s

→ measured value of time = 2.4 ± 0.1 s

(measured value of x) = $x_{best} \pm \delta x$

x_{best} = the experimenter's best estimate for the quantity concerned

he/she is reasonably confident the quantity lies somewhere between $x_{best} - \delta x$ and $x_{best} + \delta x$

δx = uncertainty/error/margin-of-error in the measurement of x

δx is defined to be positive

so that:

$x_{best} + \delta x$ is the highest value

$x_{best} - \delta x$ is the lowest

In scientific measurement, stating a range within which we are ***absolutely certain*** the measured quantity lies is hard to make

To be ***completely certain*** that the measured quantity lies within the range $x_{best} \pm \delta x$, δx is usually chosen to be too large to be useful

To avoid this, δx is chosen so that the experimenter can state with ***a certain percent confidence*** that the actual quantity lies within the range $x_{best} \pm \delta x$

will be discussed in Chapter 4...

to be continued...

Q

A student measures the length of a simple pendulum and reports his best estimate as 110 mm and the range in which the length probably lies as 108 to 112 mm. Rewrite this result in standard form!

Q

If another student reports her measurement of a current as $I = 3.05 \pm 0.03$ amps, What is the range within which I probably lies?

Q

In chapter 1, a carpenter reported his measurement of the height of a doorway by stating that his best estimate was 210 cm and that he was confident the height was between 205 and 215 cm. Rewrite this result in the standard form!

Q

A student studying the motion of a cart on an air track measures its position, velocity and acceleration at one instant, with the results shown in the following table. Rewrite these results in the standard form!

Variable	Best estimate	Probable range
Position, x	53.3	53.1 to 53.5 (cm)
Velocity, v	−13.5	−14.0 to −13.0 (cm/s)
Acceleration, a	93	90 to 96 (cm/s^2)

[2.2] Significant Figures

because the quantity δx is an estimate of an uncertainty, it should not be stated with too much precision measuring the acceleration of gravity g:

(measured g) = 9.82 ± 0.023585 m/s^2

measuring the acceleration of gravity g:

(measured g) = 9.82 ± 0.02 m/s^2

measured speed = 6051.78 ± 30 m/s

measuring the acceleration of gravity g:

(measured g) = ~~9.82 ± 0.023585 m/s^2~~

Exception:

If the leading digit in the uncertainty δx is a 1, then keeping two significant figures in δx may be better

→ $\delta x = 0.14$ is better than reducing it to $\delta x = 0.1$

measured speed = ~~6051.78 ± 30 m/s~~

Rule for Stating Uncertainties

Experimental uncertainties should almost always be rounded to <u>one significant figure</u>

Once the uncertainty in a measurement has been estimated, the significant figures in the measured value must be considered

measured speed = 6050 ± 30 m/s

Rule for Stating Answers:

The last significant figure
in any stated answer should usually be
of the same order of magnitude
(in the same decimal position)
as the uncertainty

Example:

The answer = 92.81
The uncertainty = 30

→ 90 ± 30

An electronic calculator will carry numbers with far more digit than needed

→ these numbers do not need to be rounded in the middle of calculation but in the final answers

Example:

The answer = 92.81
The uncertainty = 0.3

→ 92.8 ± 0.3

To reduce inaccuracies caused by rounding,
any numbers to be used
in subsequent calculations
should normally retain
at least one significant figure more
than is finally justified

→

It is simpler and clearer to put the answer and uncertainty in the same form

and both have the same dimensions!

Example:

The answer = 92.81
The uncertainty = 3

→ 93 ± 3

At the end of the calculations,
the final answer should be rounded
to remove these extra insignificant figures

Example:

measured charge =

$1.61 \times 10^{-19} \pm 5 \times 10^{-21}$ coulombs

Example:

measured charge =

~~$1.61 \times 10^{-19} \pm 5 \times 10^{-21}$ coulombs~~

Q

Rewrite the following results!

measured height = 5.03 ± 0.04329 m
measured time = 1.5432 ± 1 s
measured charge = $-3.21 \times 10^{-19} \pm 2.67 \times 10^{-20}$ C
measured momentum = $3.267 \times 10^{3} \pm 42$ g cm/s
measured wavelength =
0.000,000,563 ± 0.000,000,07 m

If two measurements
of the same quantity disagree...

→ there is a ***discrepancy***

Example:

measured charge =

$(1.61 \pm 0.05) \times 10^{-19}$ coulombs

Q

Rewrite the following results!

$x = 3.323 \pm 1.4$ mm
t = 1,234,567 ± 54,321 s
$\lambda = 5.33 \times 10^{-7} \pm 3.21 \times 10^{-9}$ m
r = 0.000,000,538 ± 0.000,000,03 mm

---

discrepancy

= difference between
two measured values
of the same quantity

Q

Rewrite each of the following measurements!

$v = 8.123456 \pm 0.0312$ m/s

$x = 3.1234 \times 10^{4} \pm 2$ m

$m = 5.6789 \times 10^{-7} \pm 3 \times 10^{-9}$ kg

[2.3] Discrepancy

The measurement consists of
a best estimate
and an uncertainty

↓

discrepancy = difference between the two best estimate

Example:

Student A: 15 ± 1 ohms

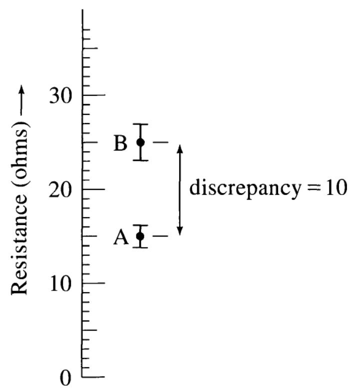
Student B: 25 ± 2 ohms

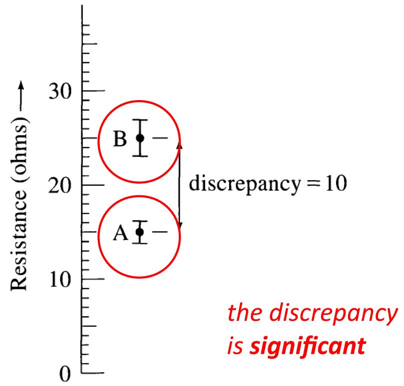
Example:

Student A: 15 ± 1 ohms

Student B: 25 ± 2 ohms

discrepancy = 25 – 15 = 10 ohms





Example:

Student C: 16 ± 8 ohms

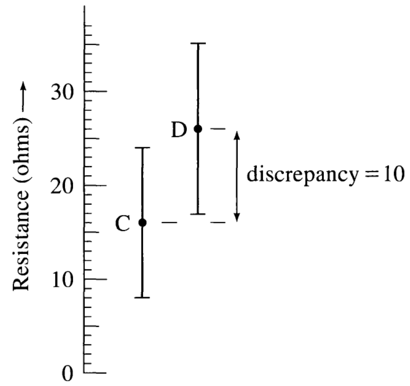
Student D: 26 ± 9 ohms

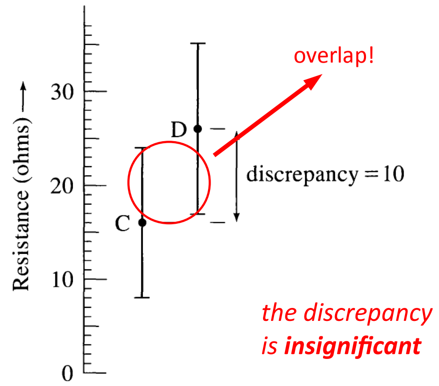
Example:

Student C: 16 ± 8 ohms

Student D: 26 ± 9 ohms

discrepancy = 26 – 16 = 10 ohms





The discrepancy between two measurements of the same quantity should be assessed not just by its size, but also by how big it is compared with the uncertainties in the measurements

Q

Two students measure the length
of the same rod and report the results
135 ± 3 mm and 137 ± 3 mm.
What is the discrepancy between
the two measurements, and is it significant?

Accepted value

= a published value of a quantity
that has been measured carefully
many times before

→ to show that they are related to one another
in accordance with some physical law

→ error analysis
is so important

Q

Each of two research groups discovers a new
elementary particle. The two reported masses are:

$m_1 = (7.8 \pm 0.1) \times 10^{-27}$ kg
and
$m_2 = (7.0 \pm 0.2) \times 10^{-27}$ kg

Based on the reported masses, would you say
they are likely to be the same particle?
If they really are, what is the discrepancy
in the two measurements?

Conclusions of an experiment which contain
a single measured number
is completely uninteresting...

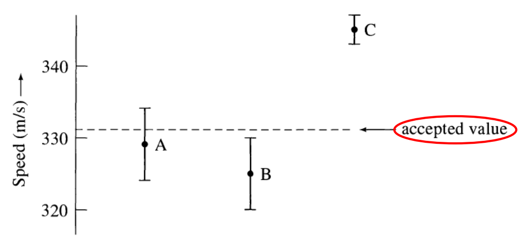
Example:

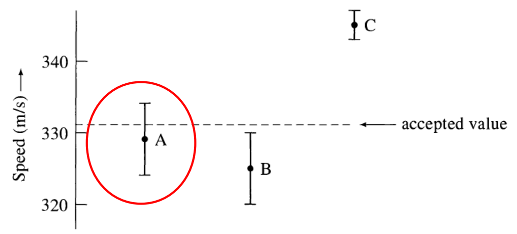
In an experiment to measure
the speed of sound in air
(at standard temperature and pressure)

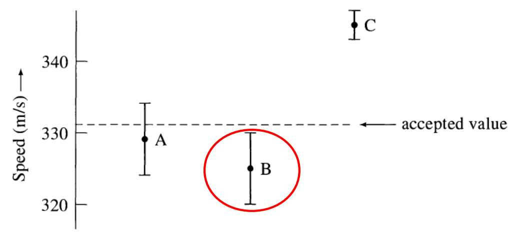
Student A's measured speed = 329 ± 5 m/s
Accepted speed = 331 m/s

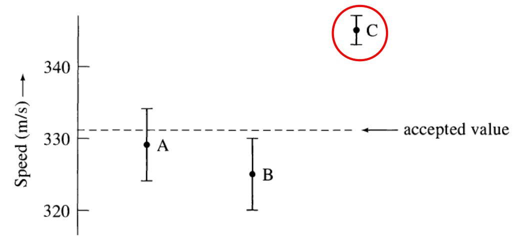
[2.4] Comparison of Measured and Accepted Values

They must <u>compare two or more numbers</u>:
a measurement with the accepted value,
a measurement with a theoretically predicted value,
or several measurements ...









A measurement can be regarded as satisfactory even if the accepted value lies slightly outside the estimated range of the measured value

If the accepted value is well outside the margins of error (the discrepancy is more than twice the uncertainty) there is reason to think something has gone wrong

Student C need to check his measurements and calculations to find out what has gone wrong:

Student B's measured speed = 325 ± 5 m/s

Student B can certainly claim that his measurement is consistent with the accepted value of 331 m/s

Student C's measured speed = 345 ± 2 m/s

Student C's discrepancy is 14 m/s (compared with the accepted speed = 331 m/s)

He may have made a mistake in the measurement or the calculation

He may have estimated his uncertainty incorrectly (345 ± **15** m/s would have been acceptable)

He may have compared his measurement with the **wrong accepted value**

The accepted value of 331 m/s is the speed of sound at standard temperature and pressure

The standard temperature is 0° C

The measurement might not be taken at standard temperature, but at room temperature

At room temperature, the speed of sound is 343 m/s

Q

A student measures the density of a liquid five times and gets the results (all in gram/cm^3) 1.8, 2.0, 2.0, 1.9 and 1.8. What is the best estimate and uncertainty based on these measurement

If the accepted value is 1.85 gram/cm^3, what is the discrepancy between the student's best estimate and the accepted value? Is it significant?

Q

Two groups of students measure the charge of the electron and report their results as follows:

Group A: $e = (1.75 \pm 0.04) \times 10^{-19}$ C
Group B: $e = (1.62 \pm 0.04) \times 10^{-19}$ C

What should each group report for the discrepancy between its value and the accepted value:

$e = 1.6 \times 10^{-19}$ C

and which of the results is satisfactory?

[2.5] Comparison of Two Measured Numbers

We want to test the law of conservation of momentum ('*the total momentum of an isolated system is constant*')

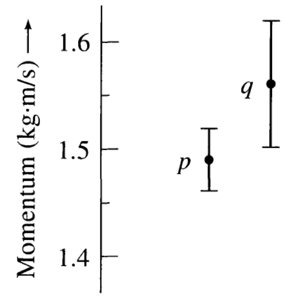
We might perform a series of experiment with two carts that collide as they move along a frictionless track

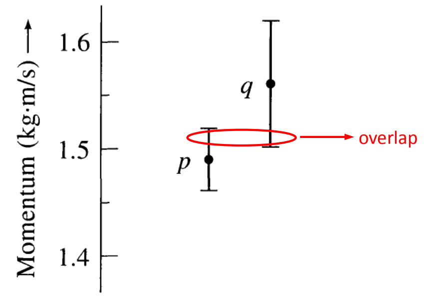
We could measure the total momentum of the two carts before (p) and after (q) they collide and check wether $p = q$ within experimental uncertainties

Example:

Initial momentum $p = 1.49 \pm 0.03$ kg m/s

Final momentum $q = 1.56 \pm 0.06$ kg m/s





The measurements are consistent with conservation of momentum

If the measurements are repeated several times, the best way to display the results:

→ use a table to record a sequence of similar measurement

→ the uncertainties often differ little from one measurement to the next

Table 2.1. Measured momenta (kg·m/s).

Trial number	Initial momentum p (all ± 0.03)	Final momentum q (all ± 0.06)
1	1.49	1.56
2	3.10	3.12
3	2.16	2.05
etc.		

For each pair of measurements, the probable range of values for p overlaps (or nearly overlaps) the range of values for q, thus the results are consistent with conservation of momentum

If we want to add another column in the table to list the difference $(p - q)$, we have to calculate the uncertainty in the difference

(measured p) = $p_{best} \pm \delta p$

(measured q) = $q_{best} \pm \delta q$

Since the best estimates for p and q are p_{best} and q_{best}, thus the best estimate for the difference $(p - q)$ is $(p_{best} - q_{best})$

To find the uncertainty in $(p - q)$, we must decide on the highest and lowest probable value of $(p - q)$

The highest value for $(p - q)$ would result if p had its largest probable value $(p_{best} + \delta p)$, at the same time that q had its smallest value $(q_{best} - \delta q)$

highest probable value = $(p_{best} - q_{best}) + (\delta p + \delta q)$

The lowest value for $(p - q)$ would result if p had its smallest probable value $(p_{best} - \delta p)$, at the same time that q had its largest value $(q_{best} + \delta q)$

lowest probable value = $(p_{best} - q_{best}) - (\delta p + \delta q)$

Thus, the uncertainty in the difference $(p - q)$ is **the sum** $\delta p + \delta q$ of the original uncertainties

Example:

$p = 1.49 \pm 0.03$ kg m/s

$q = 1.56 \pm 0.06$ kg m/s

Example:

$p = 1.49 \pm 0.03$ kg m/s

$q = 1.56 \pm 0.06$ kg m/s

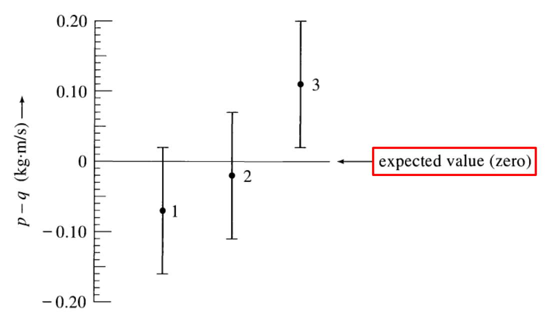
$p - q = -0.07 \pm 0.09$ kg m/s

Table 2.1. Measured momenta (kg·m/s).

Trial number	Initial momentum p (all ± 0.03)	Final momentum q (all ± 0.06)
1	1.49	1.56
2	3.10	3.12
3	2.16	2.05
etc.		

Table 2.2. Measured momenta (kg·m/s).

Trial number	Initial p (all ± 0.03)	Final q (all ± 0.06)	Difference $p - q$ (all ± 0.09)
1	1.49	1.56	−0.07
2	3.10	3.12	−0.02
3	2.16	2.05	0.11
etc.			



Uncertainty in a Difference

If two quantities x and y are measured with uncertainties δx and δy, and if the measured values x and y are used to calculate the difference $q = x - y$, **the uncertainty in q is the sum** of the uncertainties in x and y:

$$\delta q \approx \delta x + \delta y$$

This rule is called *provisional* (= temporary) because we will learn the improved rule (which is smaller) for the uncertainty in difference in Chapter 3

However, the provisional rule can be used for these reasons:

(1) it is easier to understand than the improved rule

(2) the difference between the two rules is small

(3) it always gives an upper bound (max value) on the uncertainty in $q = x - y$

This provisional rule is one of the examples of the *propagation of errors*, and it tells us how the uncertainty in x and y '*propagate*' to cause uncertainty in q

Q

In an experiment to measure the latent heat of ice, a student adds a chunk of ice to water in a styrofoam cup and observe the change in temperature as the ice melts. To determine the mass of ice added, she weighs the cup of water before and after she adds the ice and then takes the difference. If her two measurements were:

(mass of cup & water) = m_1 = 203 ± 2 grams
(mass of cup, water, & ice) = m_2 = 246 ± 3 grams

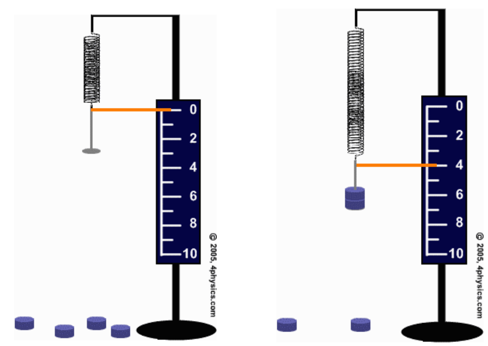
Find her answer for the mass of the ice, $m_2 - m_1$, with its uncertainty, as given by the provisional rule!

[Problem 2.9 – 2.13]

Physical Measurement Method

[Chapter 2]
Part 2

To test whether y is proportional to x,
we can plot the measured values of y
against those of x and note whether
the resulting points **do lie** on a straight line
through the origin



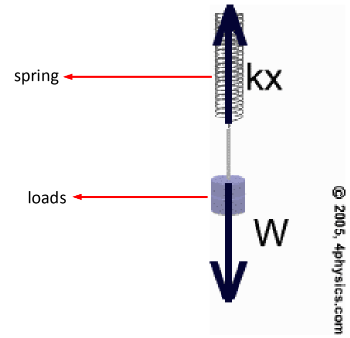
[2.6] Checking Relationships with a Graph

Example:

Hooke's Law

$F = k\,x$

force stretching a spring (F), *force constant* (k), *length of extension* (x)



If one quantity y is proportional
to some other quantity x,
a graph of y against x
is a straight line
through the origin

Robert Hooke
British Physicist
(1635 - 1705)

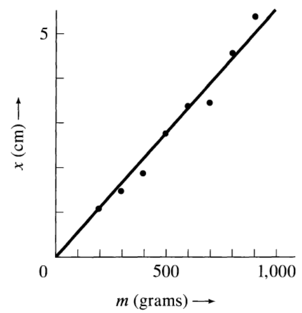
$$x = \frac{F}{k} = \frac{mg}{k} = \left(\frac{g}{k}\right)m$$

The extension x should be proportional to the load m,
and a graph of x against m should be a straight line
through the origin

If we measure x for a variety of different loads m
and plot the measured values of x and m,
the resulting points almost certainly will not lie
exactly on a straight line

Table 2.3. Load and extension.

Load m (grams) (δm negligible)	200	300	400	500	600	700	800	900
Extension x (cm) (all ± 0.3)	1.1	1.5	1.9	2.8	3.4	3.5	4.6	5.4



The points do not lie exactly on any line

Why do the points not lie exactly on any line?

- experimental uncertainties?

- mistakes we have made in the measurement?

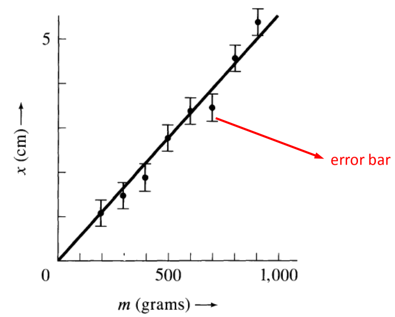
- the possibility that the extension x is not proportional to m?

The measured quantities
(extensions x and masses m)
are subject to uncertainty

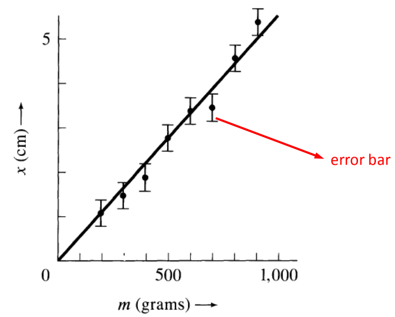
as always...

Suppose the masses used
are known very accurately,
so that the uncertainty in m is negligible

Suppose that all measurements of x
have an uncertainty
of approximately 0.3 cm



The straight line should goes through the origin
and passes through or close to all the error bars



We can conclude that the data on which the above figure is based
are consistent with x being proportional to m

From the Hooke's law equation, we see that the slope of the graph of x against m is g/k

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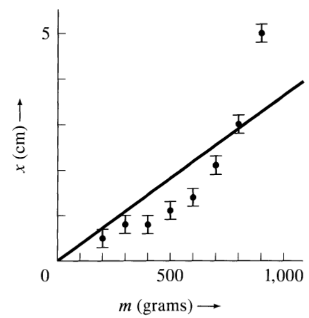
We can find the constant k of the spring by measuring the slope of the line

From the Hooke's law equation, we see that the slope of the graph of x against m is g/k

We can find the constant k of the spring by measuring the slope of the line

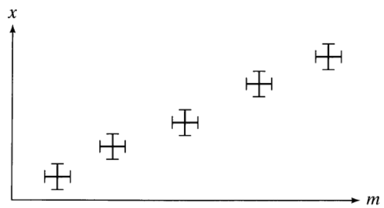
We can also find the uncertainty for k by drawing the steepest and least steep lines that fit the data reasonably well



If the best straight line misses a high proportion of the error bars or if it misses any by a large distance, our results would be *inconsistent* with x being proportional to m

If the best straight line misses a high proportion of the error bars, our results would be *inconsistent* with x being proportional to m

consider whether x is not proportional to m



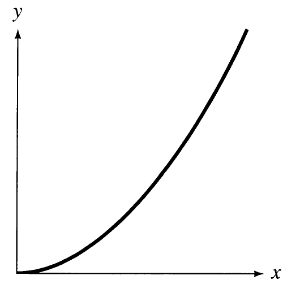
If both x and m subject to uncertainties, the simplest way to display them is to draw vertical and horizontal error bars, whose lengths show the uncertainties in x and m respectively

There is a possibility that some quantity may be expected to be proportional to a power of another

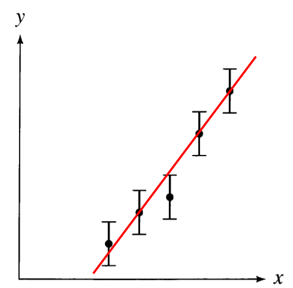
Example:

$y = A x^2$

A = constant



If the relation is $y = A x^2$, a graph of y against x should be <u>a parabola</u>



Visually judging whether a set of points fit a parabola is very hard

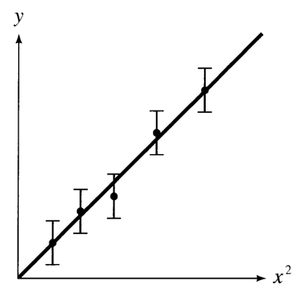
If $y = A x^n$ (where n is any power),
a graph of y against x^n should be a straight line

↓

by plotting the observed values of y against x^n,
we can check for such fit

A nongraphical way to check the proportionality
of two quantities:

If $y \propto x$, the ratio y/x should be constant



A better way to check is to plot y against x *squared*

Often one variable y depends exponentially on
another variable x

$y = A e^{Bx}$

Problem 2.14 – 2.19

If $y = A x^n$ (where n is any power),
a graph of y against x^n should be a straight line

Often one variable y depends exponentially on
another variable x

$y = A e^{Bx}$

A graph of natural logarithm of y against x
should be a straight line

[2.7] Fractional Uncertainties

The uncertainty δx in a measurement indicates the reliability or precision of the measurement

but, the uncertainty δx by itself does not tell the whole story

Uncertainty of one inch in a distance of one mil

↓

unusually precise measurement

Uncertainty of one inch in a distance of three inches

↓

crude estimate

The quality of a measurement is indicated not just by the uncertainty δx but also by the ratio of δx to x_{best}

$$\text{fractional uncertainty} = \frac{\delta x}{|x_{best}|}$$

fractional uncertainty = relative uncertainty = precision

δx = absolute uncertainty

Usually, the fractional uncertainty is represented as the *percentage uncertainty*

Example:
length l = 50 ± 1 cm

$$\text{fractional uncertainty} = \frac{\delta l}{|l_{best}|} = \frac{1\,\text{cm}}{50\,\text{cm}} = 0.02$$

Thus:
length l = 50 cm ± 2%

The fractional uncertainty is a dimensionless quantity

↓

without units

Fractional uncertainty of 10%
→ rough measurement

Fractional uncertainty of 1 ~ 2%
→ careful measurement

Fractional uncertainty of much less than 1%
→ hard to achieve

Q

Convert the errors in the following measurements of the velocities of two carts on a track into fractional errors and percent errors:

$v = 55 \pm 2$ cm/s
$u = -20 \pm 2$ cm/s

Kinetic energy of the cart:
$K = 4.58$ J $\pm$ 2%
rewrite in terms of its absolute uncertainty!

Problem 2.20 – 2.25

[2.8] Significant Figures and Fractional Uncertainties

The number of significant figures in a quantity is an approximate indicator of the fractional uncertainty in that quantity

A number with N significant figures has an uncertainty of about 1 in the N^{th} digit

$x = 21$ with two significant numbers

↓ (means)

$x = 21 \pm 1$

$y = 0.21$
(two significant figures)

↓ (means)

$y = 0.21 \pm 0.01$

...both numbers have different uncertainties but have the same fractional uncertainty

$$\frac{\delta x}{x} = \frac{\delta y}{y} = \frac{1}{21} = \frac{0.01}{0.21} = 0.05 \text{ or } 5\%$$

However, this connection is only approximate

However, this connection is only approximate

s = 10 ± 1 → 10 ± 10%

but

t = 99 ± 1 → 99 ± 1%

The fractional uncertainty associated with two significant figures ranges from 1% to 10%, depending on the first digit of the number concerned

Number of significant figures	Corresponding fractional uncertainty is	
	between	or roughly
1	10% and 100%	50%
2	1% and 10%	5%
3	0.1% and 1%	0.5%

Q

A student's calculator shows an answer 123.123.

a) If the student decides that this number actually has only three significant figures, what are its absolute and fractional uncertainties?

b) Do the same for the number 1231.23

c) Do the same for the number 321.321

d) Do the fractional uncertainties lie in the range expected for three significant figures?

Q

My calculator gives the answer x = 6.1234, but I know that x has a fractional uncertainty of 2%.

a) Restate my answer in the standard form $x_{best} \pm \delta x$ properly rounded. How many significant figures does the answer really have?

b) Do the same for y = 1.1234 with fractional uncertainty of 2%

c) Do the same for z = 9.1234

[2.9] Multiplying Two Measured Numbers

Finding momentum of a body

Finding momentum of a body

Measure its mass m and its velocity v → $p = mv$

Finding momentum of a body

Measure its mass m and its velocity $v \rightarrow$ $p = mv$

↓

Both m and v are subject to uncertainties

Finding momentum of a body

Measure its mass m and its velocity $v \rightarrow$ $p = mv$

↓

Both m and v are subject to uncertainties

↓

Uncertainty in p = ?

(measured value of x) = $x_{best} \pm \delta x$

↓

$$(\text{measured value of } x) = x_{best}\left(1 \pm \frac{\delta x}{|x_{best}|}\right)$$

For Example: $\delta x = 3\%$

↓

$$(\text{measured value of } x) = x_{best}\left(1 \pm \frac{3}{100}\right)$$

For Example: $\delta x = 3\%$

↓

$$(\text{measured value of } x) = x_{best}\left(1 \pm \frac{3}{100}\right)$$

3% uncertainty = $(0.97)\, x_{best} \leq x \leq (1.03)\, x_{best}$

$p = mv$

$$(\text{measured } m) = m_{best}\left(1 \pm \frac{\delta m}{|m_{best}|}\right)$$

$$(\text{measured } v) = v_{best}\left(1 \pm \frac{\delta v}{|v_{best}|}\right)$$

$p = mv$

(best estimate for p) = $p_{best} = m_{best} v_{best}$

$p = mv$

$$(\text{largest value for } p) = m_{best} v_{best}\left(1 \pm \frac{\delta m}{|m_{best}|}\right)\left(1 \pm \frac{\delta v}{|v_{best}|}\right)$$

---

since $\left(\frac{\delta m}{|m_{best}|}\right)\left(\frac{\delta v}{|v_{best}|}\right)$ is small number and can be neglected...

$$(\text{largest value for } p) = m_{\text{best}} v_{\text{best}} \left(1 + \frac{\delta m}{|m_{\text{best}}|} + \frac{\delta v}{|v_{\text{best}}|}\right)$$

Do the same for the smallest probable value, then we have:

$$(\text{value of } p) = m_{\text{best}} v_{\text{best}} \left(1 \pm \left[\frac{\delta m}{|m_{\text{best}}|} + \frac{\delta v}{|v_{\text{best}}|}\right]\right)$$

$$(\text{value of } p) = p_{\text{best}} \left(1 \pm \frac{\delta p}{|p_{\text{best}}|}\right)$$

$$\frac{\delta p}{|p_{\text{best}}|} = \frac{\delta m}{|m_{\text{best}}|} + \frac{\delta v}{|v_{\text{best}}|}$$

$$\frac{\delta p}{|p_{\text{best}}|} = \frac{\delta m}{|m_{\text{best}}|} + \frac{\delta v}{|v_{\text{best}}|}$$

Fractional uncertainty in *p* is the sum of the fractional uncertainties in *m* and *v*

For example:

m = 0.53 ± 0.01 kg

v = 9.1 ± 0.3 m/s

For example:

m = 0.53 ± 0.01 kg

v = 9.1 ± 0.3 m/s

$p_{\text{best}} = m_{\text{best}} v_{\text{best}}$ = (0.53)(9.1) = 4.82 kg m/s

$$\frac{\delta p}{|p_{\text{best}}|} = \frac{\delta m}{|m_{\text{best}}|} + \frac{\delta v}{|v_{\text{best}}|}$$

$$= \frac{0.01}{0.53} + \frac{0.3}{9.1} = 0.02 + 0.03 = 0.05$$

$$\delta p = \frac{\delta p}{|p_{\text{best}}|} \times p_{\text{best}} = 0.05 \times 4.82 = 0.241 \approx 0.2$$

(value of *p*) = 4.8 ± 0.2 kg m/s

Uncertainty in a product (Provisional Rule)

If x and y have fractional uncertainties $\delta x/|x_{best}|$ and $\delta y/|y_{best}|$ and if $q = xy$, then the fractional uncertainty in q is the sum of the fractional uncertainties in x and y

$$\frac{\delta q}{|q_{best}|} \approx \frac{\delta x}{|x_{best}|} + \frac{\delta y}{|y_{best}|}$$

Q

A student measures two quantities a and b and obtains the results as a = 10 ± 1 N and b = 272 ± 1 s. He now calculates the product $q = ab$. Find his answer, giving both its percent and absolute uncertainties.
Do again using a = 3.0 m ± 8% and b = 4.0 kg ± 2%

Q

To find the area of a rectangular plate, a student measures its side as l = 9.1 ± 0.1 cm and b = 3.3 ± 0.1 cm.
Express these uncertainties as percent uncertainties and then find the student's answer for the area ($A = lb$) with its uncertainty

Q

A student measures two numbers x and y as a = 10 ± 1 and b = 20 ± 1 .
What is her best estimate for their product $q = xy$?
Calculate the largest probable value of q and the smallest value of q, hence the range in which q probably lies. Compare with the provisional rule!
Do again using x = 10 ± 8 and y = 20 ± 15

→ The provisional rule was derived by assuming that the fractional uncertainties are much less than 1

Q

A student measures two quantities a and b and obtains the results as a = 11.5 ± 0.2 cm and b = 25.4 ± 0.2 s. She now calculates the product $q = ab$. Find her answer, giving both its percent and absolute uncertainties.
Do again using a = 5.0 m ± 7% and b = 3.0 N ± 1%

Physical Measurement Method

[Chapter 3]

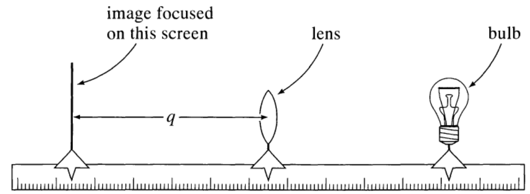
[3.1] Uncertainties in Direct Measurements

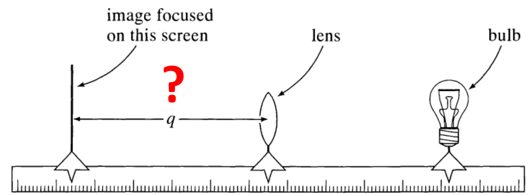
Almost all direct measurements involves reading a scale or a digital display

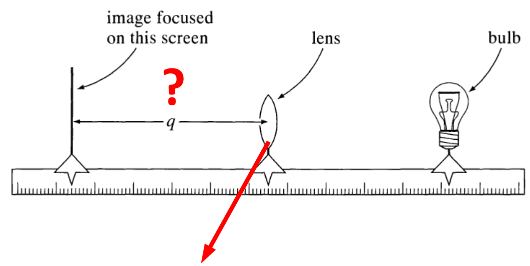
Sometimes the main sources of uncertainty are the reading of the scale and the need to interpolate between the scale markings

Unfortunately, other sources of uncertainty are frequently much more important than difficulties in scale reading

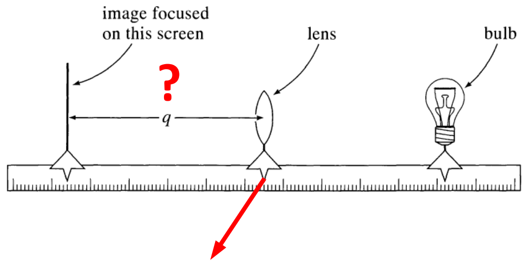
In measuring the distance between two points, our main problem may be to decide where those points really are



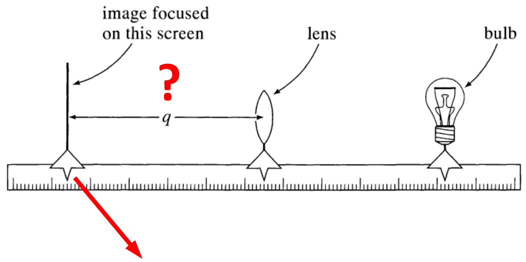




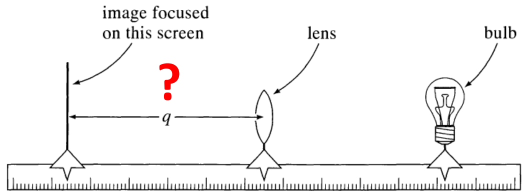
The lens is usually several millimeter thick, so locating its center is hard



If the lens comes in a bulky mounting, locating the center is even harder



The image is may appear to be well-focused throughout a range of many millimeter



Since the uncertainty arises because the two points concerned are not clearly defined, this is called a ***problem of definition***

Find all possible causes of uncertainty and estimate their effects accurately

The number of significant figures in a digital reading is the number of figures displayed

The number of significant figures in a digital reading is the number of figures displayed

→ digital voltmeter: V = 81 microvolts
thus: V = 81 ± 1 microvolts

Whenever a measurement can be repeated, it should be repeated

Whenever a measurement can be repeated, it should be repeated

→ the spread of values provides a good indication of the uncertainties

[3.2] The Square-Root Rule for a Counting Experiment

Some experiments require us
to count events that occur at random
but have a definite average rate

→ experiments in radioactivity

The uncertainty in any counted number
of random events
is the square root of the counted number

(average number of events in time T) = $\nu \pm \sqrt{\nu}$

Q

To check the activity of a radioactive sample,
an inspector places the sample in a liquid scintillation counter to count the number of decays in two-minute interval and obtains 33 counts.
What should he report as the number of decays produced by the sample in two minutes?

Q

Suppose instead, the inspector had monitored the same sample for 50 minutes and obtained 907 counts.
What would be his answer for the number of decays in 50 minutes?

→ find the percent uncertainties
in the two measurements

[3.3] Sums & Difference;
Products and Quotients

We measure two quantities x and y

(measured x) = $x_{best} \pm \delta x$

(measured y) = $y_{best} \pm \delta y$

We calculate their sum $x + y$

The highest probable value of $x + y$ is

$x_{best} + y_{best} + (\delta x + \delta y)$

The lowest probable value is

$x_{best} + y_{best} - (\delta x + \delta y)$

Thus the best estimate for $q = x + y$ is

$q_{best} = x_{best} + y_{best}$

and its uncertainty is

$\delta q = \delta x + \delta y$

Uncertainty in Sums and Differences
(Provisional Rule)

If several quantities $x, \ldots, w$ are measured with uncertainties $\delta x, \ldots, \delta w$, and the measured values used to compute

$$q = x + \cdots + z - (u + \cdots + w),$$

then the uncertainty in the computed value of q is the sum,

$$\delta q \approx \delta x + \cdots + \delta z + \delta u + \cdots + \delta w,$$

of all the original uncertainties.

If we calculate the quotient $q = x/y$

Using the fractional uncertainties:

$$(\text{value of } x) = x_{\text{best}}(1 \pm \delta x/|x|)$$

and

$$(\text{value of } y) = y_{\text{best}}(1 \pm \delta y/|y|)$$

Then

$$(\text{value of } q) = \frac{x_{\text{best}}}{y_{\text{best}}} \frac{1 \pm \delta x/|x|}{1 \pm \delta y/|y|}.$$

The largest probable value of $q = x/y$ is

$$(\text{largest value of } q) = \frac{x_{\text{best}}}{y_{\text{best}}} \frac{1 + \delta x/|x|}{1 - \delta y/|y|}.$$

Using binomial theorem:

$$\frac{1}{(1-b)} \approx 1 + b.$$

therefore

$$\frac{1+a}{1-b} \approx (1+a)(1+b) = 1 + a + b + ab$$
$$\approx 1 + a + b,$$

Thus

$$(\text{largest value of } q) = \frac{x_{\text{best}}}{y_{\text{best}}}\left(1 + \frac{\delta x}{|x|} + \frac{\delta y}{|y|}\right).$$

Using similar calculation for the smallest value:

$$(\text{value of } q) = \frac{x_{\text{best}}}{y_{\text{best}}}\left(1 \pm \left[\frac{\delta x}{|x|} + \frac{\delta y}{|y|}\right]\right).$$

Then the fractional uncertainty is:

$$\frac{\delta q}{|q|} \approx \frac{\delta x}{|x|} + \frac{\delta y}{|y|}.$$

→ when we divide or multiply two measured quantities, the uncertainty is just the sum of fractional uncertainties of both quantities

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Uncertainty in Products and Quotients (Provisional Rule)

If several quantities $x, \ldots, w$ are measured with small uncertainties $\delta x, \ldots, \delta w$, and the measured values are used to compute

$$q = \frac{x \times \cdots \times z}{u \times \cdots \times w},$$

then the fractional uncertainty in the computed value of q is the sum,

$$\frac{\delta q}{|q|} \approx \frac{\delta x}{|x|} + \cdots + \frac{\delta z}{|z|} + \frac{\delta u}{|u|} + \cdots + \frac{\delta w}{|w|},$$

of the fractional uncertainties in $x, \ldots, w$.

[3.4] Two Important Special Cases

Measured Quantity Times Exact Number

If the quantity x is measured with uncertainty δx and is used to compute the product

$$q = Bx$$

where B has no uncertainty, then the uncertainty in q is just $|B|$ times that in x

$$\delta q = |B|\delta x$$

Uncertainty in a Power

If the quantity x is measured with uncertainty δx and the measured value is used to compute the power

$$q = x^n,$$

then the fractional uncertainty in q is n times that in x,

$$\frac{\delta q}{|q|} = n\frac{\delta x}{|x|}.$$

[3.5] Independent Uncertainties in a Sum

The best estimate for $q = x + y$ is

$$q_{best} = x_{best} + y_{best}$$

and its uncertainty is

$$\delta q = \delta x + \delta y$$

The highest probable value of $x + y$ is

$$x_{best} + y_{best} + (\delta x + \delta y)$$

overestimate

It occurs if x is underestimated by δx and y is underestimated by δy

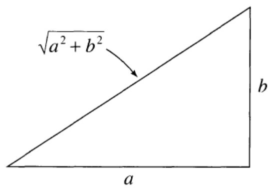
If x and y are measured independently and our errors are random in nature, we have a 50% chance that an underestimate of x is accompanied by an overestimate of y, or vice versa

→ thus $\delta q = \delta x + \delta y$ is overstate the error

If the measurements of x and y are made independently and are both governed by the normal distribution, then the uncertainty in $q = x + y$ is:

$$\delta q = \sqrt{(\delta x)^2 + (\delta y)^2}.$$

The quadrature expression is always smaller than the provisional rule



.. but there are measurements for which this cancellation is not possible

Whether or not the errors are independent and random, the uncertainty in q = x + y is <u>certainly no larger</u> than the simple sum $\delta x + \delta y$

$$\delta q \leqslant \delta x + \delta y.$$

Uncertainty in Sums and Differences

Suppose that $x, \ldots, w$ are measured with uncertainties $\delta x, \ldots, \delta w$ and the measured values used to compute

$$q = x + \cdots + z - (u + \cdots + w).$$

If the uncertainties in $x, \ldots, w$ are known to be *independent and random,* then the uncertainty in q is the quadratic sum

$$\delta q = \sqrt{(\delta x)^2 + \cdots + (\delta z)^2 + (\delta u)^2 + \cdots + (\delta w)^2}$$

of the original uncertainties. In any case, δq is never larger than their ordinary sum,

$$\delta q \leqslant \delta x + \cdots + \delta z + \delta u + \cdots + \delta w.$$

Uncertainties in Products and Quotients

Suppose that $x, \ldots, w$ are measured with uncertainties $\delta x, \ldots, \delta w$, and the measured values are used to compute

$$q = \frac{x \times \cdots \times z}{u \times \cdots \times w}.$$

If the uncertainties in $x, \ldots, w$ are *independent and random,* then the fractional uncertainty in q is the sum in quadrature of the original fractional uncertainties,

$$\frac{\delta q}{|q|} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \cdots + \left(\frac{\delta z}{z}\right)^2 + \left(\frac{\delta u}{u}\right)^2 + \cdots + \left(\frac{\delta w}{w}\right)^2}.$$

In any case, it is never larger than their ordinary sum,

$$\frac{\delta q}{|q|} \leq \frac{\delta x}{|x|} + \cdots + \frac{\delta z}{|z|} + \frac{\delta u}{|u|} + \cdots + \frac{\delta w}{|w|}.$$

The best estimate for $q(x)$ is: $q_{best} = q(x_{best})$

If δx is small, then q_{max} and q_{min} are equally spaced on either side of q_{best}

→ $q = q_{best} \pm \delta q$

[3.7] Arbitrary Functions of One Variable

The largest probable value of x is: $x_{best} \pm \delta x$

So from the graph of $q(x)$,
the largest probable value of q is q_{max}

From the graph:

$\delta q = q(x_{best} + \delta x) - q(x_{best})$

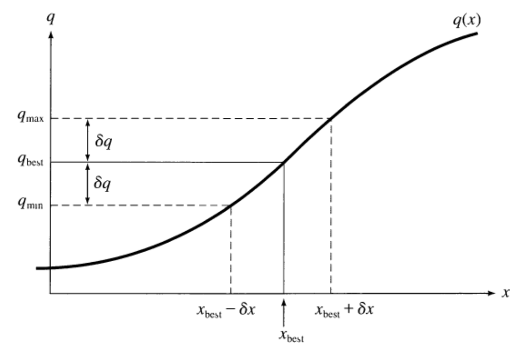
Suppose we have:

(measured x) = $x_{best} \pm \delta x$

And we want to calculate:

$q(x) = 1/\sin x$; or $q(x) = \sqrt{x}$

→ The uncertainty for $q(x)$?



Using calculus,
for any $q(x)$ with small increment u :

$$q(x + u) - q(x) = \frac{dq}{dx} u.$$

As long as the uncertainty δx is small, we have:

$$\delta q = \frac{dq}{dx}\delta x.$$

→ the uncertainty δq is the derivative dq/dx times the uncertainty δx

Uncertainty in Any Function of One Variable

If x is measured with uncertainty δx and is used to calculate the function $q(x)$, then the uncertainty δq is

$$\delta q = \left|\frac{dq}{dx}\right| \delta x$$

[3.8] Propagation Step by Step

Any errors calculation can be broken down into a sequence of steps, involving:

→ Sums & differences
→ Products & quotients
→ Computation of a function of one variable

Example:
Find the uncertainty for q:

$q = x(y - z \sin u)$

→ compute the uncertainty in the function $\sin u$
→ then for the product of z and $\sin u$
→ next in the difference of y and $z \sin u$
→ finally for the product of x and $x(y - z \sin u)$

The stepwise method can only be applied for independent variables

Example:

$q = y - x \sin y$

→ add their uncertainties directly, not in quadrature
→ using stepwise method gives bigger uncertainty for q

[3.11] General Formula for Error Propagation

$$q = \frac{x + y}{x + z}$$

dependent variables

If we overestimate x,
we overestimate both $x + y$ and $x + z$

and these overestimate may cancel one another
when we calculate $(x + y)/(x + z)$
→ *compensating errors*

→ can not be seen using stepwise method

We measure two quantities x and y

(measured x) = $x_{best} \pm \delta x$

(measured y) = $y_{best} \pm \delta y$

We calculate some function $q = q(x, y)$

For one variable:

If the extreme probable value of x are

$x_{best} \pm \delta x$

then

$q(x_{best} \pm \delta x)$

Using the approximation:

$$q(x + u) \approx q(x) + \frac{dq}{dx} u$$

then

$$q(x_{best}) \pm \left| \frac{dq}{dx} \right| \delta x,$$

For two variables:

$$q_{best} = q(x_{best}, y_{best})$$

then:

$$q(x + u, y + v) \approx q(x, y) + \frac{\partial q}{\partial x} u + \frac{\partial q}{\partial y} v,$$

Thus:

$$q(x_{best}, y_{best}) \pm \left(\left| \frac{\partial q}{\partial x} \right| \delta x + \left| \frac{\partial q}{\partial y} \right| \delta y \right).$$

Or:

$$\delta q \approx \left| \frac{\partial q}{\partial x} \right| \delta x + \left| \frac{\partial q}{\partial y} \right| \delta y.$$

Uncertainty in a Function of Several Variables

Suppose that $x, \ldots, z$ are measured with uncertainties $\delta x, \ldots, \delta z$ and the measured values are used to compute the function $q(x, \ldots, z)$. If the uncertainties in $x, \ldots, z$ are independent and random, then the uncertainty in q is

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x} \delta x\right)^2 + \cdots + \left(\frac{\partial q}{\partial z} \delta z\right)^2}.$$

In any case, it is never larger than the ordinary sum

$$\delta q \leq \left| \frac{\partial q}{\partial x} \right| \delta x + \cdots + \left| \frac{\partial q}{\partial z} \right| \delta z.$$