CHAPTER 1

Axioms of Probability

1.1. Random Experiments

In simple terms, a **random experiment** (or experiment) is a process whose outcome is uncertain. It is often useful to think of this process as being repeatable but, in practice, this is seldom the case. This "conceptual repeatability" is important as it allows us to interpret probabilities in terms of long-run frequencies.

For a given random experiment, the following terms are defined:

- The **sample space** is the set of all possible outcomes of a random experiment. We will denote the sample space by Ω. If a sample space Ω is either finite or countably infinite, then it is called a **discrete** sample space. A sample space for outcomes which value in some interval of real numbers is called **continuous** sample space.
- A subset of the sample space Ω is called an **event**. An event A has occured if the true outcome of the experiment is contained in A ($\omega \in A$). An event consisting of no outcomes is called the empty set and will be denoted by ϕ.

EXAMPLE 1. Following are some examples:

- If the experiment consists of tossing two dice, then the sample space consists of the 36 points.

  $$\Omega = \{(i, j) : i, j = 1, 2, ..., 6\}$$

  where the outcome (i, j) is said to occur if i appears on the left most die and j on the other die
- If the experiment consists of measuring (in hours) the lifetime of a transistor, then the sample space consists of all nonnegative real numbers; that is,$\Omega = \{x : 0 \leq x \leq \infty\}$.

Let A and B be arbitrary events defined on a sample space Ω .

- The union of A and B ($A \cup B$) consists of all outcomes that belong to at least one of A and B. That is, $\omega \in A \cup B$ if, and only if, $\omega \in A$ or $\omega \in B$.
- The intersection of A and B ($A \cap B$) consists of all outcomes that belong to both A and B. That is, $\omega \in A \cap B$ if, and only if, $\omega \in A$ and $\omega \in B$.
- The complement of A (A^c) consists of all outcomes in Ω that do not belong to A. That is, $\omega \in A^c$ if, and only if, $\omega \notin A$. A and B are disjoint (or mutually exclusive) if $A \cap B = \phi$.
- The operations of forming unions, intersections, and complements of events obey certain rules similar to the rules of algebra.

  Commutative laws: $A \cup B = B \cup A$ and $A \cap B = B \cap A$

  Associative laws: $(A \cup B) \cup C = A \cup (B \cup C)$ dan $(A \cap B) \cap C = A \cap (B \cap C)$

Distributive laws: $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ dan $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

DeMorgan's laws: $\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c$ dan $\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c$

1.2. Probability Measures

One possible approach to assigning probabilities to events involves the notation of relative frequency. If $m(A)$ represents the number of times that the event A occurs among M trials of a given experiment, then $f_A = \frac{m(A)}{M}$ represents the **relative frequency** of occurrence of A on these trials of the experiment. If, for any event A, the limit of f_A as M approaches infinity exists, then one could assign probability A to by,

$$P(A) = \lim_{M \to \infty} f_A$$

Although the relative frequency approach may not always be adequate as a practical method of assigning probabilities, it is the way that probability usually is interpreted. However, many people consider this interpretation too restrictive.

Given a random experiment with a sample space Ω, **a function or measure** $P(.)$ is defined on the subsets (events) of Ω that assigns a real number to each event; this number will represent the probability that a given event occurs.

DEFINITION 2. $P(.)$ is called a probability measure if the following axioms are satisfied:

(1) $P(A) \geq 0$ for any event A
(2) $P(\Omega) = 1$
(3) If $A_1, A_2, ...$ are disjoint events then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{n} P(A_i)$

A number of simple but useful properties of probability measures can be derived from the **Definition** 2 given above.

PROPOSITION 3. *The following are consequence of the axioms of probability:*

(1) $P(A^c) = 1 - P(A)$
(2) For any event A, $P(A) \leq 1$
(3) For any two event A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(4) For any three event A, B, and C, $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap B) - P(A \cap B) + P(A \cap B \cap C)$
(5) If $A \subset B$, then $P(A) \leq P(B)$
(6) **Boole's Inequality** Let $A_1, A_2, ...$ be any events. Then $P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$
(7) **Bonferroni's Inequality** Let $A_1, A_2, ...$ be any events. Then $P\left(\bigcap_{i=1}^{\infty} A_i\right) \geq 1 - \sum_{i=1}^{n} P(A_i^c)$

In many problem, including those involving games of chance, the nature of the outcomes dictates the assignment of equal probability to each elementary event. This type of model sometimes is referred to as the **classical probability model**. Let the sample space consist of N distinct outcomes,

$$S = \{e_1, e_2, ..., e_N\}$$

The equally likely assumption requires the values p_i that

$$p_1 = p_2 = ... = p_N$$

and, to satisfy equation $p_i \geq 0, \forall i$ and $\sum_i p_i = 1$, necessarily

$$p_i = P(\{e_i\}) = \frac{1}{N}$$

In this case, because all terms in the sum $P(A) = \sum_{e_i \in A} P(\{e_i\})$ are the same, it follows that $P(A) = \frac{n(A)}{N}$.

1.2.1. Exercises.

(1) Suppose that A and B are mutually exclusive events for which $P(A) = 0.3$ and $P(B) = 0.5$ .What is the probability that
  (a) either A or B occurs?
  (b) A occurs but B does not?
  (c) both A and B occur

(2) A retail establishment accepts either the American Express or the VISA credit card. A total of 24 percent of its customers carry an American Express card, 61 percent carry a VISA card, and 11 percent carry both cards. What percentage of its customers carry a credit card that the establishment will accept?

(3) Sixty percent of the students at a certain school wear neither a ring nor a necklace. Twenty percent wear a ring and 30 percent wear a necklace. If one of the students is chosen randomly, what is the probability that this student is wearing
  (a) a ring or a necklace?
  (b) a ring and a necklace?

(4) A total of 28 percent of American males smoke cigarettes, 7 percent smoke cigars, and 5 percent smoke both cigars and cigarettes.
  (a) What percentage of males smokes neither cigars nor cigarettes?
  (b) What percentage smokes cigars but not cigarettes?

(5) An elementary school is offering 3 language classes: one in Spanish, one in French, and one in German. The classes are open to any of the 100 students in the school. There are 28 students in the Spanish class, 26 in the French class, and 16 in the German class. There are 12 students that are in both Spanish and French, 4 that are in both Spanish and German, and 6 that are in both French and German. In addition, there are 2 students taking all 3 classes.
  (a) If a student is chosen randomly, what is the probability that he or she is not in any of the language classes?
  (b) If a student is chosen randomly, what is the probability that he or she is taking exactly one language class?

(6) A certain town with a population of 100,000 has 3 newspapers: I, II, and III. The proportions of townspeople who read these papers are as follows: I: 10 percent, I and II: 8 percent, I and II and III: 1 percent, II: 30 percent, I and III: 2 percent, III: 5 percent, II and III: 4 percent. (The list tells us, for instance, that 8000 people read newspapers I and II.)
   (a) Find the number of people who read only one newspaper.
   (b) How many people read at least two newspapers?
   (c) If I and III are morning papers and II is an evening paper, how many people read at least one morning paper plus an evening paper?
   (d) How many people do not read any newspapers?
   (e) How many people read only one morning paper and one evening paper?

(7) If it is assumed that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of being dealt
   (a) a flush? (A hand is said to be a flush if all 5 cards are of the same suit.)
   (b) one pair? (This occurs when the cards have denominations a, a, b, c, d,where a, b, c,and d are all distinct.)
   (c) two pairs? (This occurs when the cards have denominations a, a, b, b, c,where a, b,and c are all distinct.)
   (d) three of a kind? (This occurs when the cards have denominations a, a, a, b, c,where a, b, and c are all distinct.)
   (e) four of a kind? (This occurs when the cards have denominations a, a, a, a, b.)

(8) Two cards are randomly selected from an ordinary playing deck. What is the probability that they form a blackjack? That is, what is the probability that one of the cards is an ace and the other one is either a ten, a jack, a queen, or a king?

(9) Two symmetric dice have both had two of their sides painted red, two painted black, one painted yellow, and the other painted white. When this pair of dice is rolled, what is the probability that both dice land with the same color face up?

(10) A pair of fair dice is rolled. What is the probability that the second die lands on a higher value than does the first?

(11) If two dice are rolled, what is the probability that the sum of the upturned faces equals i? Find it for i = 2, 3, ... , 11, 12. 25.

(12) An urn contains 3 red and 7 black balls. Players A and B withdraw balls from the urn consecutively until a red ball is selected. Find the probability that A selects the red ball. (A draws the first ball, then B, and so on. There is no replacement of the balls drawn.)

(13) An urn contains 5 red, 6 blue, and 8 green balls. If a set of 3 balls is randomly selected, what is the probability that each of the balls will be
   (a) of the same color?
   (b) of different colors? Repeat under the assumption that whenever a ball is selected, its color is noted and it is then replaced in the urn before the next selection. This is known as sampling with replacement

(14) Two cards are chosen at random from a deck of 52 playing cards.What is the probability that they
   (a) are both aces?
   (b) have the same value?
(15) Show that the probability that exactly one of the events E or F occurs equals $P(E) + P(F) - 2P(EF)$.
(16) Prove that $P(EF^c) = P(E) - P(EF)$.