CHAPTER 1

Random Variable

Consider an experiment where three coins is tossed. For such experiments, there are many event that can be defined on Ω, but only those events involving is considered, for example, the number of heads that appear. In this case, the set of all possible values of the number of heads can be defined taking on one of the values 0, 1, 2, and 3 with respective probabilities

$$P\{Y=0\} = P\{(T,T,T)\} = \frac{1}{8}$$

$$P\{Y=1\} = P\{(H,T,T),(T,H,T),(T,T,H)\} = \frac{3}{8}$$

$$P\{Y=2\} = P\{(H,H,T),(H,T,H),(T,H,H)\} = \frac{3}{8}$$

$$P\{Y=3\} = P\{(H,H,H)\} = \frac{1}{8}$$

Consider another experiment of selecting three balls randomly without replacement from an urn containing 20 balls numbered 1 through 20. A bet is put for at least one of the balls that are drawn has a number as large as or larger than 17, what event that is considered and its probability? These events from two examples above can be associated to random varible.

Definition 1. A random variable X is a function that maps the sample space to the real line; that is, for each $\omega \in \Omega$, $X(\omega)$ is a real number.

There are two types of random variables based its sample space, discrete random variable and continuous random variable.

1.1. Discrete Random Variables

Definition 2. If the set of all possible values of a random variable, X, is a countable set, $x_1, x_2, ..., x_n$ or $x_1, x_2, ...$, then X is called a **discrete random variable**. The function

$$f(x) = P(X=x) \qquad x = x_1, x_2, ...$$

that assigns the probability to each possible value x will be called the discrete probability density function (discrete pdf), probability mass function, probability function or density function.

Theorem 3. *A function $f(x)$ is a discrete pdf if and only if it satisfies both of the following properties for at most a countably infinite set of reals $x_1, x_2, ...$*

$$f(x_i) \geq 0$$

for all x_i*, and*

$$\sum_{all\, x_i} f(x_i) = 1$$

PROOF. From Definition ??. □

DEFINITION 4. The cummulative distribution function (CDF) of a random variable X is defined for any real x by

$$F(x) = P(X \leq x)$$

The general relationship between $F(x)$ and $f(x)$ for a discrete distribution is given by the following theorem.

THEOREM 5. *Let* X *be a discrete random variable with pdf* $f(x)$ *and CDF* $F(x)$*. If the possible values of* X *are indexed in increasing order,* $x_1 < x_2 < x_3 ...$*, then* $f(x_1) = F(x_1)$ *and for any* $i > 1$*,*

$$f(x_i) = F(x_i) - F(x_{i-1})$$

Furthermore, if $x < x_1$ *then* $F(x) = 0$*, and for any other real* x

$$F(x) = \sum_{x_i \leq x} f(x_i)$$

where the summation is taken over all indices i *such that* $x_i \leq x$

The CDF of any random variable must satisfy the properties of the following theorem.

THEOREM 6. *A function* $F(x)$ *is a CDF for some random variable* X *if and only if it satisfies the following properties:*

$$\lim_{x \to -\infty} F(x) = 0 \tag{1.1.1}$$

$$\lim_{x \to \infty} F(x) = 1 \tag{1.1.2}$$

$$\lim_{h \to 0^+} F(x+h) = F(x) \tag{1.1.3}$$

$$a < b\, implies\, F(a) \leq F(b) \tag{1.1.4}$$

Property (1.1.3) says that $F(x)$ is continuous from the right. And property (1.1.4) says that $F(x)$ is nondecreasing.

Some important properties of probability distribution involve numerical quantities called expected values.

DEFINITION 7. If X is a discrete random variable with pdf $f(x)$, then the expected value of X is defined by

$$E(X) = \sum_x x f(x)$$

The mean or expected value of a random variable is a "weight average," and it can be considered as a measure of the "center" of the associated probability distribution.

1.2. Continuous Random Variables

Each work day a man rides a bus to his place of business. His waiting time on any given morning to be a random variable X. Suppose that he is very observant and noticed over the years that the frequency of days when he waits no more than x minutes for the bus is proportional to x for all x.

$$P(X \leq x) = ...$$

And a new bus arrives promptly every five minutes.

$$... \leq x \leq ...$$

$$P(... \leq X \leq ...) = ...$$

DEFINITION 8. A random variable X is called a continuous random variable if there is a function $f(x)$ called the probability density function (pdf) of X, such that the CDF can be represented as

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

Above definition provides a way to derive the CDF when the pdf is given, specifically,

The function $F(x)$, as represented in Definition 8, is unaffected regerdless of how such values is treated. Thus, $P(X = c) = 0$ and $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$.

THEOREM 9. *A function $f(x)$is a pdf for some continuous random variable X if and only if*

Other properties of probability distribution can be described in terms of quantities called percentiles.

DEFINITION 10. Under this assumption, the inverse function F^{-1} is well defined; $x = F^{-1}(y)$ if $y = F(x)$. The pth quantile of the distribution F is defined to be that value x_p such that $F(x_p) = p$, or $P(X < x_p) = p$.

Special cases are $p = \frac{1}{2}$, which corresponds to the median of F, and $p = \frac{1}{4}$ and $p = \frac{3}{4}$, which correspond to the lower and upper quartiles of F.

DEFINITION 11. If X is a continuous random variable with pdf $f(x)$, then the expected values of X is defined by

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

Some properties of expected value of a function of X are useful to consider.

THEOREM 12. $E(aX + b) = ...$

DEFINITION 13. The variance of a random variable X is given by

$$Var\left(X\right)=E\left[\left(X-E\left(X\right)\right)^{2}\right]=...$$

$Var\left(aX\right)=...$
$Var\left(aX+b\right)=...$
Certain special expected values, called moments, are useful in characterizing some features of the distribution.

DEFINITION 14. The kth moment about the origin of a random variables is

$$\mu_{k}=E\left(X\right)^{k}$$

and the kth moment about the mean is

$$\mu_{k}=E\left[X-E\left(X\right)\right]^{k}=E\left(X-\mu\right)^{k}$$

A special expected value that is quite useful is known as the moment generating function.

DEFINITION 15. If X is a random variables, then the expected value

$$M_{X}\left(t\right)=E\left(e^{tX}\right)$$

is called the moment generating function (MGF) of X if this expected value exists for all values of t in some interval of $-h<t<h$ the form for some $h>0$

THEOREM 16. $M_{X}^{(r)}\left(0\right)=E\left(X^{r}\right)$

PROOF. ... □

THEOREM 17. If $Y=aX+b$ then $M_{Y}\left(t\right)=...$

PROOF. ... □

1.3. Problems

(1) A discrete random variable X has a pdf $f_{X}(x)=c\left(8-x\right),x=0,1,2,3,4,5$ and zero otherwise.
  (a) Find c
  (b) Find $F\left(x\right)$
  (c) Find $P\left(X>2\right)$
  (d) Find $E\left(X\right)$
(2) Suppose that X is a continuous random variable whose probability density function is given by

$$f_{X}(x)=\begin{cases} C\left(4x-2x^{2}\right) & ,0<x<2 \\ 0 & ,otherwise\end{cases}$$

  (a) What is the value of C?
  (b) Find $P\left(X>1\right)$
  (c) Find $F\left(x\right)$
(3) A random variable X has the pdf

$$f_{X}(x)=\begin{cases} x^{2} & ,0<x\leq 1 \\ \frac{2}{3} & ,1<x\leq 2 \\ 0 & ,otherwise\end{cases}$$

   (a) Find the median of X?
   (b) Sketch the graph of the CDF and show the position of the median on the graph
(4) Find median of X with CDF $F(x) = 1 - e^{-\left(\frac{x}{3}\right)^2}, x > 0$
(5) Find mode of X if pdf of X is $f(x) = \frac{x}{10}, x = 2, 3, 5$
(6) Find mode of $X \sim f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-2)^2}, -\infty < x < \infty$
(7) Let X be a random variable with discrete pdf $f(x) = \frac{x}{8}, x = 1, 2, 5$, and zero otherwise. Find
   (a) $E(X)$
   (b) $Var(X)$
   (c) $E(2x + 3)$
(8) Let X be a continuous random variable with pdf $f(x) = 3x^2, 0 < x < 1$, and zero otherwise. Find
   (a) $E(X)$
   (b) $Var(X)$
   (c) $E(X^r)$
   (d) $E\left(3X - 5X^2 + 1\right)$
(9) Suppose that X is a random variable with MGF $M_x(t) = \frac{1}{8}e^t + \frac{1}{4}e^{2t} + \frac{5}{8}e^{5t}$
   (a) What is the distribution of X
   (b) What is $P[X = 2]$
(10) Assume that X be a continuous random variable with discrete pdf $f(x) = e^{-(x+2)}, -2 < x < \infty$, and zero otherwise. Find
   (a) Find moment generating function of X
   (b) Use the MGF of (a) to find $E(X)$ and $E\left(X^2\right)$