CHAPTER 1

Variable Random Transformation

Suppose X is a random variable and $Y = g(X)$ for some function g. Given the distribution of X, what is the distribution of Y?

1.1. One-To-One Transformation

Let $g(x)$ be a real-valued function of a real variable x. If the equation $y = g(x)$ can be solved uniquely, say $x = h(y)$ then the transformation is one-to-one.

THEOREM 1. *Suppose that X is a discrete random variable with pdf $f_X(x)$ and that $Y = g(X)$ defines a one-to-one transformation. In other words, the equation $y = g(x)$ can be solved uniquely, say $x = h(y)$. Then the pdf of Y is*

$$f_Y(y) = f_X(h(y)), y \in B \tag{1.1.1}$$

where $B = \{y \mid f_Y(y) > 0\}$.

PROOF. This follow because

$$f_Y(y) = P[Y = y] = P[g(X) = y] = P[X = h(y)] = f_X(h(y))$$

□

EXAMPLE 2. Let $X \sim GEO(p)$, so that $f_X(x) = pq^{x-1}, x = 1, 2, 3, \ldots$. Let $Y = X - 1$, then $g(x) = x - 1$, $h(y) = y + 1$, and $f_Y(y) = pq^{(y+1)-1}, y = 0, 1, 2, \ldots$.

THEOREM 3. *Suppose that X is a continuous random variable with pdf $f_X(x)$ and assume that $Y = g(X)$ defines a one-to-one transformation from $A = \{x \mid f_X(x) > 0\}$ on to $B = \{y \mid f_Y(y) > 0\}$ with inverse transformation $x = h(y)$. If the derivative $\frac{dh(y)}{dy}$ is continuous and non zero on B, then the pdf of Y is*

$$f_Y(y) = f_X(h(y)) \left| \frac{dh(y)}{dy} \right|, y \in B \tag{1.1.2}$$

PROOF. If $y = g(x)$ is one-to-one transformation, then it is either monotonic increasing and monotonic decreasing. If it is assumed that it is increasing, then $g(x) \le y$ if and only if $x \le h(y)$, and thus

$$F_Y(y) = P[Y \le y] = P[g(X) \le y] = P[X \le h(y)] = F_X(h(y))$$

and consequently,

$$f_Y(y) = \frac{dF_X(h(y))}{dy} = \frac{dF_X(h(y))}{dh(y)} \frac{dh(y)}{dy} = f_X(h(y)) \left| \frac{dh(y)}{dy} \right|$$

because $g'(x) > 0$ when $g(x)$ is increasing then $\frac{dh(y)}{dy} > 0$.

In the decreasing case, $g(x) \leq y$ if and only if $h(y) \leq x$, and thus

$$F_Y(y) = P[Y \leq y] = P[g(X) \leq y] = P[X \geq h(y)] = 1 - F_X(h(y))$$

and,

$$f_Y(y) = -f_X(h(y))\frac{dh(y)}{dy} = f_X(h(y))\left|\frac{dh(y)}{dy}\right|$$

because $g'(x) < 0$ when $g(x)$ is decreasing then $\frac{dh(y)}{dy} < 0$. □

The derivative of $h(y)$ is usually referred to as the **Jacobian** of the transformation, and denoted by $J = \frac{dh(y)}{dy}$.

EXAMPLE 4. Suppose $F_X(x) = 1 - e^{-2x}, 0 < x < \infty$, determine the pdf of $Y = e^X$.

Solution. The inverse transformation is $x = h(y) = \ln y$, and the Jacobian is $J = h'(y)) = \frac{1}{y}$, so that

$$f_Y(y) = f_X(\ln y)\left|\frac{1}{y}\right| = 2e^{-2(\ln y)}\left(\frac{1}{y}\right) = 2y^{-3}, 1 < y < \infty$$

EXAMPLE 5. Suppose $Y \sim LOGN(\mu, \sigma^2)$, determine the pdf of $X = \ln Y$.

Solution. The pdf of Y is

$$f_Y(y) = \frac{1}{y\sigma\sqrt{2\pi}}e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}}, 0 < y < \infty$$

with parameter $-\infty < \mu < \infty$ and $0 < \sigma < \infty$. The inverse transformation is $y = g(x) = e^x$, and the Jacobian is $J = g'(x)) = e^x$, so that

$$f_X(x) = f_Y(e^x)|e^x| = \frac{1}{e^x\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}e^x = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$$

Therefore $X \sim N(\mu, \sigma^2)$.

EXAMPLE 6. Suppose $X \sim N(0,1)$, define the pdf of $Y = \mu + \sigma X$ for some $\sigma > 0$.

Solution. The pdf of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, -\infty < x < \infty$$

The inverse transformation is $x = h(y) = \frac{y-\mu}{\sigma}$, and the Jacobian is $J = h'(y)) = \frac{1}{\sigma}$, so that

$$f_Y(y) = f_X(\frac{y-\mu}{\sigma})\left|\frac{1}{\sigma}\right| = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(y-\mu)^2}{2\sigma^2}}, -\infty < y < \infty$$

Therefore $Y \sim N(\mu, \sigma^2)$.

EXERCISE 7. Work the following exercise

(1) Suppose $f_X(x) = \frac{x}{8}, x = 1, 2, 5$, determine the pdf of $Y = 3X + 4$
(2) Let X have $f_X(x) = \frac{3}{4}(1 - x^2), -1 < x < 1$, determine the pdf of $Y = X + 2$
(3) Suppose $X \sim N(\mu, \sigma^2)$ and $0 < \sigma < \infty$, determine the pdf of $Z = \frac{X-\mu}{\sigma}$

(4) Let X have $f_X(x) = \frac{x^2}{24}, -2 < x < 4$, determine the pdf of $Y = X^2$

THEOREM 8. *If X is continuous with CDF $F(X)$, then $U = F(X) \sim UNIF(0,1)$.*

PROOF. It is assumed that $F(x)$ is one-to-one transformation then $F^{-1}(u)$ is existed.

$$F_U(u) = P[U \leq u] = P[F(X) \leq u] = P\left[X \leq F^{-1}(u)\right] = F\left(F^{-1}(u)\right) = u$$

Since $0 \leq F(x) \leq 1$, then $F_U(u) = 0$ if $u \leq 0$ and $F_U(u) = 1$ if $u \geq 1$. Therefore,

$$F_U(u) = \begin{cases} 0, & 0 \geq u \\ u, & 0 < u < 1 \\ 1, & u \geq 1 \end{cases}$$

□