0.1. Not One-To-One Transformation

Suppose that the function of $u(x)$is not one-to-one transformation over $A = \{x \mid f_X(x) > 0\}$. It is possible to partition A into disjoint subsets $A_1, A_2, ...$ such that is one-to-one over each A_j. Then for each y in the range of $g(x)$, the equation $y = g(x)$ has a unique solution $x_j = h_j(y)$ over the set A_j. It follows that Theorem **??** and **??** can be extended to functions that are not one-to-one by replacing equations **??** and **??** with

$$f_Y(y) = \sum_j f_X(h_j(y)) \tag{0.1.1}$$

for the discrete case and,

$$f_Y(y) = \sum_j f_X(h_j(y)) \left| \frac{dh_j(y)}{dy} \right| \tag{0.1.2}$$

for the continuous case.

EXAMPLE 1. Let $f_X(x) = \frac{4}{31}\left(\frac{1}{2}\right)^x, x = -2, -1, 0, 1, 2$ and consider $Y = |X|$.

Solution. Clearly, $B = \{0, 1, 2\}$and
$f_Y(0) = f_X(0) = \frac{4}{31}$
$f_Y(1) = f_X(-1) + f_X(1) = \frac{8}{31} + \frac{2}{31} = \frac{10}{31}$
$f_Y(2) = f_X(-2) + f_X(2) = \frac{16}{31} + \frac{1}{31} = \frac{17}{31}$
Another way to express this is
$f_Y(0) = \frac{4}{31}, y = 0$
$f_Y(1) = \frac{4}{31}\left[\left(\frac{1}{2}\right)^{-y} + \left(\frac{1}{2}\right)^{y}\right], y = 1, 2$

EXAMPLE 2. Suppose $X \sim UNIF(-1, 1)$ and $Y = X^2$. Determine pdf of Y.

It is possible to partition $A = (-1, 1)$ into disjoint subsets $A_1 = (-1, 0)$ and $A_2 = (0, 1)$. Since A is continuous then $x = 0$ can be neglected. Then for each y in the range of $g(x)$, the equation $y = g(x)$ has a unique solution $x_1 = h_1(y) = -\sqrt{y}$ over the set A_1 and $x_2 = h_2(y) = \sqrt{y}$ over the set A_2. Thus the pdf of Y is

$$f_Y(y) = f_X\left(-\sqrt{y}\right)\left|\frac{-1}{2\sqrt{y}}\right| + f_X\left(\sqrt{y}\right)\left|\frac{1}{2\sqrt{y}}\right| = \frac{1}{2\sqrt{y}}, \ y \in (0, 1)$$

EXAMPLE 3. Let $f_X(x) = \frac{x^2}{3}, -1 < x < 2$ and consider $Y = X^2$. Determine pdf of Y.

There are two inverse transformation, $x_1 = h_1(y) = -\sqrt{y}$ for $x < 0$ and $x_2 = h_2(y) = \sqrt{y}$ for $x > 0$. Thus the pdf of Y is

$$f_Y(y) = \begin{cases} f_X\left(-\sqrt{y}\right)\left|\frac{-1}{2\sqrt{y}}\right| + f_X\left(\sqrt{y}\right)\left|\frac{1}{2\sqrt{y}}\right| = \frac{1}{2\sqrt{y}}\left[\frac{\left(-\sqrt{y}\right)^2}{3} + \frac{\left(\sqrt{y}\right)^2}{3}\right] & , 0 < y < 1 \\ f_X\left(\sqrt{y}\right)\left|\frac{-1}{2\sqrt{y}}\right| = \frac{1}{2\sqrt{y}}\left[\frac{\left(-\sqrt{y}\right)^2}{3}\right] & , 1 < y < 4 \end{cases}$$