0.1. Sums of Random Variables

Sums of independent random variables often arise in practice. In examples, determine the distribution of a sum of $S = X_1 + X_2$ where X_1 and X_2 are continuous random variables.

EXAMPLE 1. Let X_1 and X_2 be independent and uniform, $X_i \sim UNIF\,(0, 1)$. Consider the transformation $T = X_1$ and $S = X_1 + X_2$. Find the pdf of S.

Solution.
The pdf of X is

$$f_S(s) = \begin{cases} s & ,0 < s < 1 \\ 2 - s & ,1 \le x < 2 \end{cases}$$

EXAMPLE 2. Let $f_U\,(u) = e^{-u}, u > 0$ and $f_V\,(v) = 2v, 0 < v < 1$. U and V are independent. If $X = U + V$, then the pdf of X is?

Solution.

(1) Consider the transformation $X = U + V$ and $Y = V$. The pdf of X is

$$f_X(x) = \begin{cases} {}_0\int^x 2ye^{-(x-y)}dy = ... & ,0 < x < 1 \\ {}_0\int^1 2ye^{-(x-y)}dy = ... & ,1 \le x < \infty \end{cases}$$

(2) Consider the transformation $X = U + V$ and $Y = U$. The pdf of X is

$$f_X(x) = \begin{cases} {}_0\int^x 2\,(x-y)\,e^{-y}dy = ... & ,0 < x < 1 \\ {}_{x-1}\int^x 2\,(x-y)\,e^{-y}dy = ... & ,1 \le x < \infty \end{cases}$$

The pdf of X is

$$f_X(x) = \begin{cases} 2x + 2e^{-x} - 2 & ,0 < x < 1 \\ 2e^{-x} & ,x \ge 1 \end{cases}$$

EXAMPLE 3. Let X_1 and X_2 be independent and uniform, $X_i \sim UNIF\,(0,\,1)$. Consider the transformation $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = X_1.X_2$. Find the pdf of Y_1 and Y_2.

Solution.

EXAMPLE 4. Let X_1 and X_2 are independent gamma variables, $f\,(x_1,\,x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x_1^{\alpha-1} x_2^{\beta-1} e^{-x_1-x_2}, 0 < x_i < \infty$. Consider the transformation $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1+X_2}$.

The joint pdf of Y_1 and Y_2 is

$$f_{Y_1,Y_2}(y_1, y_2) = ..., (y_1, y_2) \in ...$$

And the pdf of Y_1 is

EXAMPLE 5. Let X_1, X_2 and X_3 are independent gamma variables, $X_i \sim GAM\,(1, \alpha_i)\,, i = 1, 2, 3$. Consider the transformation $Y_i = \frac{X_i}{\sum_{j=1}^{3} X_j}, i = 1, 2$ and

$Y_3 = \sum_{j=1}^{3} X_j$.

The joint pdf of Y_1, Y_2, and Y_3 is

$$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = ... \,, \; (y_1, y_2, y_3) \in ...$$

And the pdf of Y_3 is

A technique based on moment generating functions usually is much more convenient than using transformations for determining the distribution of sums of independent random variables.

THEOREM 6. *If $X_1, X_2, ..., X_n$ are independent random variables with MGFs $M_{X_i}(t)$ then the MGF of $Y = \sum_{i=1}^{n} X_i$ is*

$$M_Y(t) = M_{X_1}(t) \ldots M_{X_n}(t)$$

PROOF. Notice that $e^{tY} = e^{t(X_1+\ldots+X_n)} = e^{tX_1}\ldots e^{tX_1}$ so
$M_Y(t) = E\left(e^{tY}\right) = E\left(e^{t(X_1+\ldots+X_n)}\right) = E\left(e^{tX_1}\right)\ldots E\left(e^{tX_1}\right) = M_{X_1}(t)\ldots M_{X_n}(t)$ □

EXAMPLE 7. Let $X_1, X_2, ..., X_k$ be independent binomial random variables with respective parameters n_i and p, $X_i \sim BIN(n_i, p)$, and let $Y = \sum_{i=1}^{k} X_i$.

It follows that $M_Y(t) = ...$
Thus, $Y \sim ...$

EXAMPLE 8. Let $X_1, X_2, ..., X_n$ be independent Poisson-distributed random variables with respective parameters n_i and p, $X_i \sim POI(\mu_i)$, and let $Y = \sum_{i=1}^{n} X_i$.

It follows that $M_Y(t) = ...$
Thus, $Y \sim ...$

EXAMPLE 9. Let $X_1, X_2, ..., X_n$ be independent Gamma-distributed with respective shape parameter $\kappa_1, \kappa_2, ..., \kappa_n$ and common scale parameter θ, $X_i \sim GAM(\theta, \kappa_i)$ for $i = 1, 2, ..., n$, and let $Y = \sum_{i=1}^{n} X_i$.

It follows that $M_Y(t) = ...$
Thus, $X \sim ...$

EXAMPLE 10. Let $X_1, X_2, ..., X_n$ be independent normally distributed random variables, $X_i \sim N\left(\mu_i, \sigma_i^2\right)$, and let $Y = \sum_{i=1}^{n} X_i$.

It follows that $M_Y(t) = ...$
Thus, $Y \sim ...$

This includes the special case of a random sample $X_1, X_2, ..., X_n$ from a normally distributed population, say $X_i \sim N\left(\mu, \sigma^2\right)$. In this case, $\mu = \mu_i$ and $\sigma^2 = \sigma_i^2$ for all $i = 1, 2, ..., n$, and consequently $\sum_{i=1}^{n} X_i \sim N\left(n\mu, n\sigma^2\right)$. It also follows readily in the case that the sample mean, $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ is normally distributed, $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$.