CHAPTER 1

Converges in Probability

It is possible, in some cases, to find bounds on probabilities based on moments.

PROPOSITION 1. *(Markov's inequality) Let X be a random variable, then for any value $t > 0$,*

$$P(X \geq a) \leq \frac{E(X)}{a}$$

PROOF. For $a > 0$, let

$$I = \begin{cases} 1 & , X \geq a \\ 0 & , otherwise \end{cases}$$

since $X \geq 0$ then $I \leq \frac{X}{a}$.

Taking expectations of the above yields that $E(I) \leq \frac{E(X)}{a}$. $E(I) = P(X \geq a)$, therefore

$$P(X \geq a) \leq \frac{E(X)}{a}$$

□

As a corollary, Proposition 2 can be obtained.

PROPOSITION 2. *(Chebyshev's inequality) Let X be a random variable with mean μ and variance σ^2. Then, for any $t > 0$,*

$$P(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}$$

PROOF. For the continuous case (the discrete case is entirely analogous), let $R = \{x : |x - \mu| > t\}$ then

$$P(|X - \mu| \geq t) \leq \int_R f(x)\,dx$$

if $x \in R$,

$$\frac{|x - \mu|^2}{t^2} \geq 1$$

Thus,

$$\int_R f(x)\,dx \leq \int_R \frac{|x - \mu|^2}{t^2} f(x)\,dx \leq \int_{-\infty}^{\infty} \frac{(x - \mu)^2}{t^2} f(x)\,dx = \frac{\sigma^2}{t^2}$$

□

For another interpretation, set $t = k\sigma$ so that the inequality becomes

$$P\left(|X-\mu| \geq k\sigma\right) \leq \frac{1}{k^2} \tag{1.0.1}$$

and an alternative form is

$$P\left(|X-\mu| < k\sigma\right) \geq 1 - \frac{1}{k^2} \tag{1.0.2}$$

Chebyshev's inequality has the following consequence.

COROLLARY 3. *If $Var(X) = 0$, then $P(X = \mu) = 1$*

PROOF. If $P(X = \mu) < 1$, then for some $\varepsilon > 0$, $P\left(|X-\mu| \geq \varepsilon\right) > 0$. However, by Chebyshev's inequality, for any $\varepsilon > 0$,

$$P\left(|X-\mu| \geq \varepsilon\right) = 0$$

Contradiction, $P(X = \mu) = 1$ □

DEFINITION 4. (Convergence in Probability) The sequence of random variables Y_n is said to **converge in probability to** Y , written $Y_n \rightarrow_p Y$, if

$$\lim_{n\to\infty} P(|Y_n - Y| < \varepsilon) = 1$$

THEOREM 5. *The sequence of random variables $Y_1, Y_2, ...$ is said to **convergence stochastically** to a constant c, written $Y_n \rightarrow_{stochastic} c$, if and only if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} P(|Y_n - c| < \varepsilon) = 1$$

The sequence of random variables that satisfies Theorem 5 is also said to **converge in probability** to a constant c, written $Y_n \rightarrow_p c$.

EXAMPLE 6. Let $X_1, X_2, ..., X_n$ be a random sample from a uniform distribution, $X_i \sim BIN\left(1, \pi\right)$ and let $Y_n = \frac{\sum_{i=1}^{n} X_i}{n}$. Show that $Y_n \rightarrow_p \pi$.

Solution. Using Proposition 2,

$$P(|Y_n - \pi| < k\sqrt{\frac{\pi\left(1-\pi\right)}{n}}) \geq 1 - \frac{1}{k^2}$$

Choose $\varepsilon = k\sqrt{\frac{\pi(1-\pi)}{n}})$, then

$$\lim_{n\to\infty} P(|Y_n - \pi| < \varepsilon) \geq \lim_{n\to\infty}\left(1 - \frac{\pi\left(1-\pi\right)}{\varepsilon^2 n}\right)$$

Therefore,

$$\lim_{n\to\infty} P(|Y_n - \pi| < \varepsilon) = 1$$

THEOREM 7. ***(Strong Law of Large Numbers)*** *If $X_1, X_2, ..., X_n$ is a random sample from a distribution with finite mean μ and varianceσ^2, then the sequence of sample means convergence in probability to μ or $\lim_{n\to\infty} P(|\bar{X}_n - \mu| < \varepsilon) = 1$, written $\bar{X}_n \rightarrow_{sp} \mu$.*

PROOF. First, find $E(\bar{X_n})$ and $Var(\bar{X_n})$, $E(\bar{X_n}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \mu$. Since the X_i are independent, then $Var(\bar{X_n}) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{\sigma^2}{n}$. The desired result now follows immediately from Proposition 2 (Chebyshev's inequality), which states that

$$\lim_{n\to\infty} P(|\bar{X_n} - \mu| < \varepsilon) \geq \lim_{n\to\infty} 1 - \frac{Var(\bar{X_n})}{\varepsilon^2} = \lim_{n\to\infty} 1 - \frac{\sigma^2}{n\varepsilon^2} = 1$$

□

THEOREM 8. *(**Weak Law of Large Numbers**) If $X_1, X_2, ..., X_n$ is a random sample from a distribution with finite mean μ and varianceσ^2, then the sequence of sample means convergence in probability to μ or $\lim_{n\to\infty} P(|\bar{X_n} - \mu| > \varepsilon) = 0$, written $\bar{X}_n \to_{wp} \mu$.*

PROOF. First, find $E(\bar{X_n})$ and $Var(\bar{X_n})$, $E(\bar{X_n}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \mu$. Since the X_i are independent, then $Var(\bar{X_n}) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{\sigma^2}{n}$. The desired result now follows immediately from Proposition 2 (Chebyshev's inequality), which states that

$$\lim_{n\to\infty} P(|\bar{X_n} - \mu| > \varepsilon) \leq \lim_{n\to\infty} \frac{Var(\bar{X_n})}{\varepsilon^2} = \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0$$

□

THEOREM 9. *If $Z_n = \frac{\sqrt{n}(Y_n - m)}{c} \to_d Z \sim N(0,1)$ then $Y_n \to_p m$*

PROOF. If $Z_n = \frac{\sqrt{n}(Y_n - m)}{c} \to_d Z \sim N(0,1)$, then from Theorem **??** $Y_n \to_d N\left(m, \frac{c^2}{n}\right)$. Using Proposition 2,

$$P(|Y_n - m| < k\frac{c}{\sqrt{n}}) \geq 1 - \frac{1}{k^2}$$

Choose $\varepsilon = k\frac{c}{\sqrt{n}}$, then $k = \frac{\varepsilon\sqrt{n}}{c}$ and

$$\lim_{n\to\infty} P(|Y_n - m| < \varepsilon) \geq \lim_{n\to\infty} \left(1 - \frac{c^2}{\varepsilon^2 n}\right)$$

Therefore,

$$\lim_{n\to\infty} P(|Y_n - m| < \varepsilon) = 1$$

□

THEOREM 10. *For a sequence of random variables, if $Y_n \to_p Y$ then $Y_n \to_d Y$*

PROOF. ... □

THEOREM 11. *If $Y_n \to_p c$ then for any function $g(y)$ that is continuous at c,*

$$g(Y_n) \to_p g(c)$$

PROOF. ... □

THEOREM 12. *If X_n and Y_n are two sequence of random variables such that $X_n \to_p c$ and $Y_n \to_p d$ then,*

(1) $aX_n + bY_n \to_p ac + bd$

(2) $X_nY_n \to_p cd$
(3) $\frac{X_n}{c} \to_p 1, c \neq 0$
(4) $\frac{1}{X_n} \to_p \frac{1}{c}, \forall n\, P[X_n \neq 0] = 1, c \neq 0$
(5) $\sqrt{X_n} \to_p \sqrt{c}, \forall n\, P[X_n \geq 0] = 1$

PROOF. ... □

THEOREM 13. *(Slutsky's Theorem) If X_n and Y_n are two sequence of random variables such that $X_n \to_p c$ and $Y_n \to_d Y$ then,*

(1) $X_n + Y_n \to_d c + Y$
(2) $X_nY_n \to_d cY$
(3) $\frac{Y_n}{X_n} \to_d \frac{Y}{c}, c \neq 0$

PROOF. (As a special case X_n could be an ordinary numerical sequence such as $X_n = \frac{n}{(n-1)}$)... □

THEOREM 14. *If $Y_n \to_d Y$ then for any continuous function $g(y)$, $g(Y_n) \to_d g(Y)$*

PROOF. (Assume $g(y)$ is not to depend on n) ... □

THEOREM 15. *If $\frac{\sqrt{n}(Y_n - m)}{c} \to_d Z \sim N(0,1)$ and if $g(y)$ has nonzero derivative at $y = m$, $g'(m) \neq 0$, then*

$$\frac{\sqrt{n}\left[g(Y_n) - g(m)\right]}{|cg'(m)|} \to_d Z \sim N(0,1)$$

PROOF. ... □

1.1. Problems

(1) Let $X_1, X_2, ..., X_n$ be a random sample from a uniform distribution, $X_i \sim BIN(1, \pi)$ and let $Y_n = \frac{\sum_{i=1}^{n}(X_i - \pi)}{n}$. Show that $Y_n \to_p 0$.