HANDOUT

DIFFERENTIAL EQUATIONS

For International Class

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CONTENTS

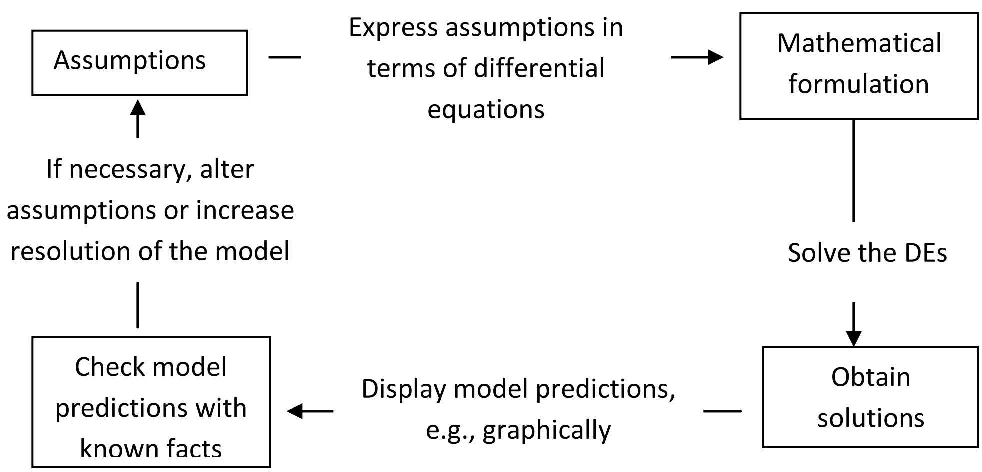
STATE UNIVERSITY OF-YOGYAKARTA
FACULTY OF-MATHEMATICS AND NATURAL SCIENCE
DEPARTMENT OF MATHEMATICS EDUCATION

First Meeting : I. Introduction to Differential Equations
- Some Basic Mathematical Models
- Definitions and Terminology

The study of differential equations has attracted the attention of many of the world's greatest mathematicians during the past three centuries. To give perspective to your study of differential equations, first, we use a problem to illustrate some of the basic ideas that we will return to and elaborate upon frequently throughout the remainder book.

1.1 Some Basic Mathematical Models

Consider the diagram below.



Example 1.1

The population of the city of Yogyakarta increases at a rate proportional to the number of its inhabitants present at any time t. If the population of Yogyakarta was 30,000 in 1970 and 35,000 in 1980, what will be the population of Yogyakarta in 1990?

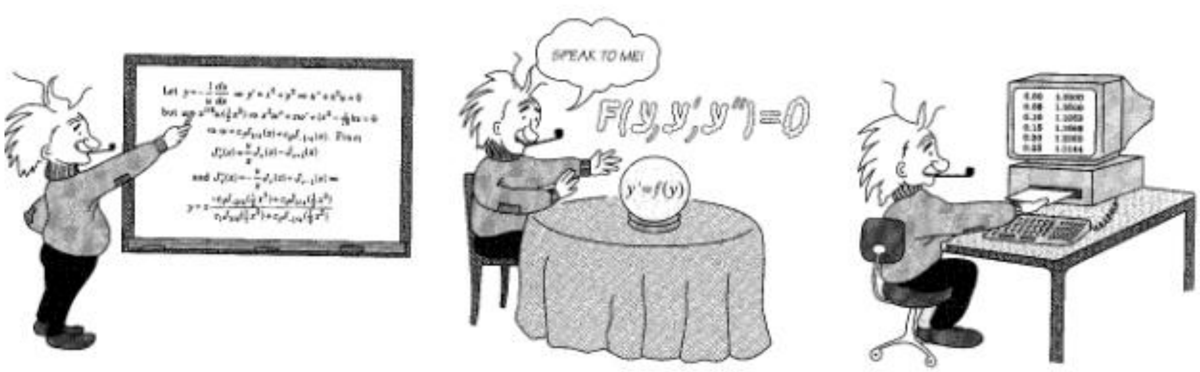
We can use the theory of differential equation to solve this problem. Suppose $y(t)$ defines the number of population at time-*t*. Therefore, the first information can be transformed into mathematical model as

$$\frac{dy}{dt} = Ky$$

for some constant *K*, $y(1970) = 30.000$, and $y(1980) = 35.000$.

Just what is a differential equation and what does it signify? Where and how do differential equations originate and of what use are they? Confronted with a differential equation, what does one do with it, how does one do it, and what are the results of such activity? These questions indicate three major aspects of the subject: theory, method, and application. Therefore, things that will be discussed in this handout are how the forms of differential equations are, the method to find its solution for each form and the last one is its applications.

When we talk about something that is constantly changing, for example rate, then I can say, in truth, we talk about derivative. Many of the principles underlying the behavior of the natural world are statement or relations involving rates at which things happen. When expressed in mathematical terms the relations are equations and the rates are derivatives. Equations containing derivatives are differential equations.



(a) analytical **(b)** qualitative **(c)** numerical

1.2 Definitions and Terminology

Definition **Differential Equation**

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a differential equation (DE).

For examples

1. $\frac{dy}{dx} = 2x$
2. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 8y = 0$
3. $\left(\frac{dy}{dx}\right)^2 - 2y = 0$
4. $(x^2 + y^2)dx + (y - x)dy = 0$
5. $x\frac{dy}{dx} + (x + 1)y = x^3$

Other examples of differential equation are

1. **Radioactive Decay.** The rate at which the nuclei of a substance decay is proportional to the amount (more precisely, the number of nuclei) $A(t)$ of the substance remaining at time t.

$$\frac{dA}{dt} = kA$$

2. **Newton's Law of Cooling/warming**. Suppose $T(t)$ represents the temperature of a body at time *t*, Tm the temperature of the surrounding medium. The rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium.

$$\frac{dT}{dt} = k(T - T_m)$$

More examples can be found at page 20 – 24, A first course in Differential Equation with modeling application, Dennis G. Zill.

Exercises. I

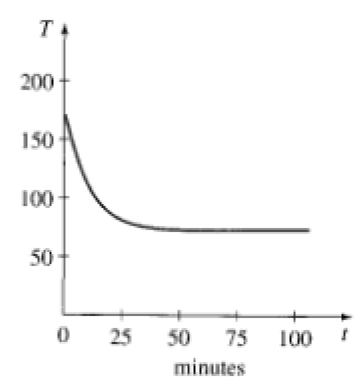
Population Dynamics.

1. Assumed that the growth rate of a country at a certain time is proportional to the total population of the country at that time $P(t)$. Determine a differential equation for the population of a country when individuals are allowed to immigrate into the country at a constant rate $r > 0$. How if individuals are allowed to emigrate from the country at a constant rate $r > 0$.
2. In another model of a changing population of a community, it is assumed that the rate at which the population changes is a net rate – that is, the difference between the rate of births and the rate of deaths in the community. Determine a model for the population $P(t)$ if both the birth rate and the death rate are proportional to the population present at time t.
3. Using the concept of net rate introduced in Problem 2, determine a model for a population $P(t)$ if the birth rate is proportional to the population present at time t but the death rate is proportional to the square of the population present at time t.
4. Modify the model in problem 3 for the net rate at which the population $P(t)$ of a certain kind of fish changes by also assuming that the fish are harvested at a constant rate $r > 0$.

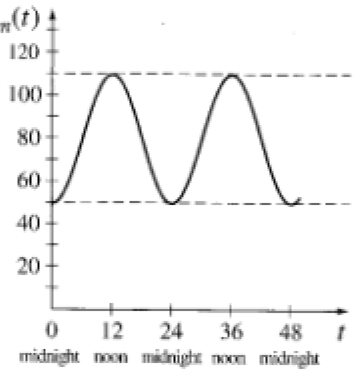
Newton's Law of Cooling/Warming

5. A cup of coffee cools according to Newtion's Law of cooling. Use data from the graph of the temperature $T(t)$ in figure beside to estimate the constants T_m, T_0 and k in a model of the form of a first-order initial-value problem

$$\frac{dT}{dt} = k(T - T_m), T(0) = T_0$$



6. The ambient temperature T_m could be a function of time t. Suppose that in an artificially controlled environment, T_m is periodic with a 24-hour period as illustrated in Figure beside. Devise a mathematical model for the temperature $T(t)$ of a body within this environment.



Spread of a Disease/Technology

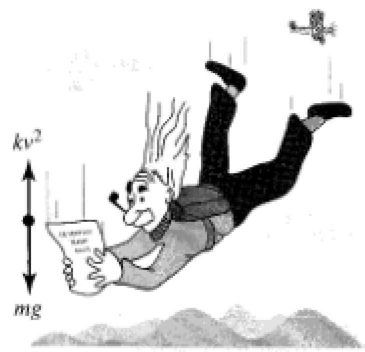
7. Suppose a student carrying in a flu virus returns to an isolated college campus of 1000 students. Determine a differential equation for the number of people $x(t)$ who have contracted the flu if the rate at which the disease spreads is proportional to the number of interaction between the number of students who have the flu and the number of number of students who have the flu and the number of students who have not yet been exposed to it.

Mixtures

8. Supposes that a large mixing tank initially holds 300 gallons of water in which 50 pounds of salt have been dissolved. Pure water is pumped into the tank at a rate of 3 gal/min, and when the solution is well stirred, it is then pumped out at the same rate. Determine the differential equation for amount of salt $A(t)$ in the time at time t. What is $A(0)$.
9. Suppose that a large mixing tank initially holds 300 gallons of water in which 50 pounds of salt have been dissolved. Another brine solution is pumped into the tank at a rate of 3 gal/min, and when the solution is well stirred, it is then pumped out at a slower rate 2 gal/min. If the concentration of the solution entering is 2 lb/gal, determine a differential equation for the amount of salt $A(t)$ in the tank at time t.

Falling bodies and Air Resistance

10. For high-speed motion through the air – such as the skydiver shown in figure below, falling before the paracute is opened – air resistance is closer to a power of the instantaneous velocity $v(t)$. Determine a differential equation for the velocity $v(t)$ of a falling body of mass m if air resistance is proportional to the square of the instantaneous velocity.



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Second Meeting : I. Introduction to Differential Equations
- Classification of Differential Equations
- Initial Value Problems
- Boundary Value Problems

Differential equation is divided into two big classes; those are ordinary differential equation and partial differential equation. The different is located at the number of its independent variable.

Definition **Ordinary Differential Equation**

A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation (ODE).

Definition **Partial Differential Equation**

A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a partial differential equation (PDE)

Definition **Order**

The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation

Definition **Linear Ordinary Differential Equation**

A linear ordinary differential equation of order *n*, in the dependent variable *y* and the independent variable *x*, is an equation that is in, or can be expressed in, the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x)$$

Where a_0 is not identically zero.

Definition **Nonlinear Ordinary Differential Equation**

A nonlinear ordinary differential equation is an ordinary differential equation that is not linear.

As given from the previous materials, there are many terminologies for differential equations. Among those terminologies, differential equations are then classified by type, order and linearity.

Classification by Type

Refers to the number of its independent variables, differential equation is divided into two types, ordinary differential equation and partial differential equation.

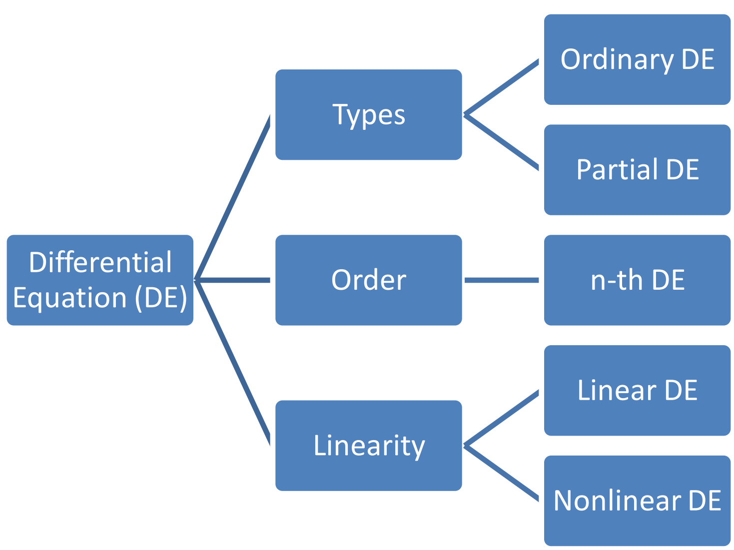
Classification by Order

Consider to the highest derivative in the equation, differential equation can be classified by *n*th-order ordinary differential equations.

Classification by Linearity

From its linearity, differential equation is classified into linear differential equation and nonlinear differential equation.

Classification of Differential Equation can be easily seen from the diagram below:



For example

1. $\frac{dy}{dx} = 2y$

   → First order homogeneous linear ordinary differential equation

2. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 8y = 0$

   → Third order homogeneous linear ordinary differential equation

3. $\left(\frac{d^2y}{dx^2}\right)^3 - 2y = 0$

   → Second order homogeneous nonlinear ordinary differential equation

4. $(x^2 + y^2)dx + (y - x)dy = 0$

   → first order

5. $x\frac{dy}{dx} + (x + 1)y = x^3$

   → first order non homogeneous ordinary differential equation

Suppose that you are driving along a road and you want to predict where you will be in the future. You need to know two things--

- How fast you are driving.
- Where you start.

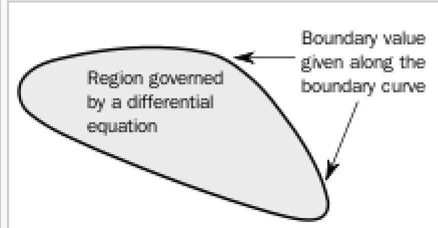
Now suppose that you are the mayor of a city and want to predict how many students will be enrolled the schools in the future. Once again you need to know two things--

- How fast the school population is changing.
- The current school population.

The two parts of each of the problems above together make up an **initial value problem** or **IVP**. An **IVP** has two parts--

- A differential equation describing how things change.
- One or more **initial conditions** describing how things start.

Definition of Initial Value Problems



Shows a region where a differential equation is valid and the associated boundary values

On some interval I containing x_0, the problem

Solve :

$$\frac{d^n y}{dx^n} = f\left(x, y, y', \dots, y^{(n-1)}\right)$$

Subject to : $y(x_0) = y_0,$

$$y'(x_0) = y1, \dots, y^{n-1}(x_0) = y_{n-1}$$

Where $y_0, y_1, y_2, \dots y_{n-1}$ are arbitrarily specified real constants, is called an initial value problem (IVP). The given values of the unknown function and its first derivatives at a single point x_0, $y(x_0) = y_0,\ y'(x_0) = y1, \dots, y^{n-1}(x_0) = y_{n-1}$ are called initial conditions.

Another type of problem consists of solving a linear differential equation in which the dependent variable y or its derivatives are specified at different points. This type of differential equation is called boundary value problem (BVP)

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Third Meeting : I. Introduction to Differential Equations
- Autonomous Equation

An important class of first order equations is those in which the independent variable does not appear explicitly. Such equations are called autonomous and have the form

$$\frac{dy}{dt} = f(y)$$

For example

$$\frac{dy}{dt} = ay + b,$$

where a and b are constants. Our goal now is to obtain important qualitative information directly from the differential equation, without solving the equation.

Steps to draw the solution curves of $\frac{dy}{dt} = f(y)$.

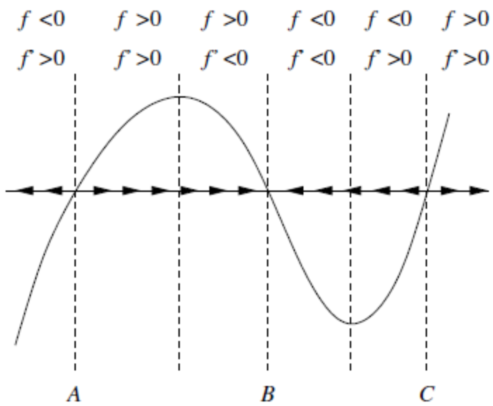
1. Find the equilibrium solutions, those are constant solutions or solutions which satisfying $\frac{dy}{dt} = 0$.
2. Determine which area such that the curve will increase or decrease.

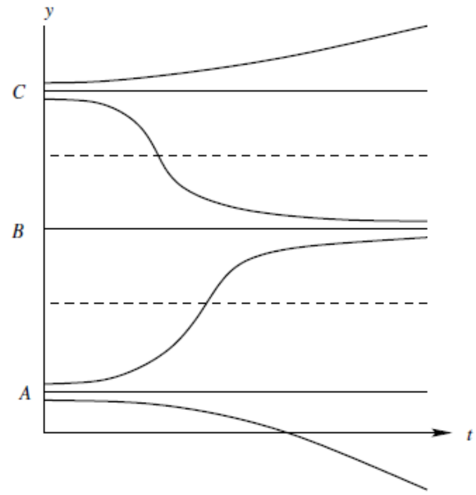
   The curve will increase for area where $\frac{dy}{dt} > 0$ and decrease for area where $\frac{dy}{dt} < 0$
3. To investigate one step further, we can determine the concavity of the solution curve by using its second derivative, $\frac{d^2y}{dt^2}$

   $$\frac{d^2y}{dt^2} = f'(y)\frac{dy}{dt} = f'(y)f(y)$$

   The graph of y versus t is concave up when $\frac{d^2y}{dt^2} > 0$ while concave down when $\frac{d^2y}{dt^2} < 0$.

Look at the following figures.





Example 1.3

The population of the city of Yogyakarta increases at a rate proportional to the number of its inhabitants present at any time t. The population of Yogyakarta was 30,000 in 1970 and 35,000 in 1980. Determine the behavior of the number of its population.

Answer :

As we modeled this problem before, suppose $y(t)$ defines the number of population at time-*t*. Therefore, the first information can be transformed into mathematical model as

$$\frac{dy}{dt} = Ky$$

for some constant *K* > *0*, $y(1970) = 30.000$, and $y(1980) = 35.000$.

From the concept of derivative, we know that first derivative of a function is the rate. As we see that k > 0 and y > 0 which are implied that $\frac{dy}{dt} > 0$. Therefore we can simply conclude that the number of population will increase all over the time.

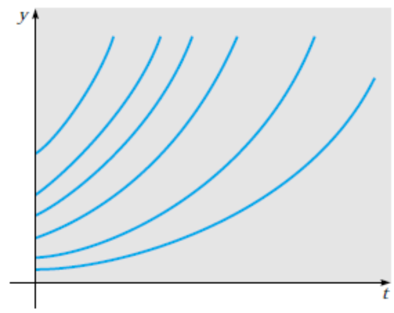


Figure : *y* versus *t* for *dy*/*dt* = *Ky*.

Example 1.4

If the population rate can be modeled into

$$\frac{dy}{dt} = (3 - y)y$$

Answer :

1. The equilibrium solutions are y such that
$$(3 - y)y \leftrightarrow y = 3 \ \text{ or } \ y = 0$$
2. Here, we can analyze that for $0 < y < 3$ the derivative is positive valued while for $y < 0$ or $y > 3$ the derivative is negative valued. Therefore, the population will increase for $0 < y < 3$, while for $y < 0$ or $y > 3$ the population will decrease.
3. Determine the concavity.
$$\frac{d^2y}{dt^2} = f'(y)\frac{dy}{dt} = f'(y)f(y)$$

Suppose $f(y) = (3 - y)y$, then $f'(y) = -2y + 3$. It's implied that $\frac{d^2y}{dt^2} = (3 - 2y)(3 - y)y$.

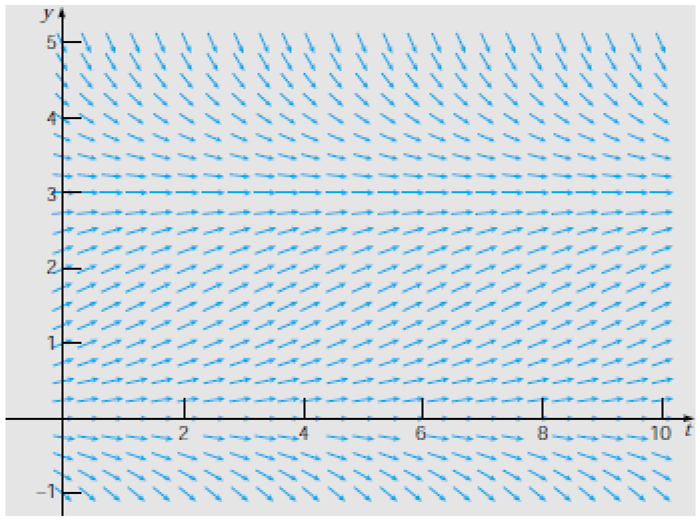


Figure : y versus t for $\frac{dy}{dt} = (3 - y)y$

Definition

Consider the value for the solution y that starts near the equilibrium solution y_0 as $t \to \infty$. If

$$\lim_{t \to \infty} y(t) = y_0$$

then y_0 is called an asymptotically stable solution. Otherwise, it's called an unstable equilibrium solution.

Example I.4 above, $y = 3$ is an asymptotically stable solution whereas $y = 0$ is an unstable equilrium.

Exercises 3.

In each problem below, sketch the graph of y versus t, determine the critical (equilibrium) points, and classify each one as asymptotically stable or unstable.

1. $\frac{dy}{dt} = 2y + 3y^2, y(0) = 3$
2. $\frac{dy}{dt} = y(y-1)(y-2), y(0) = 4$
3. $\frac{dy}{dt} = y(y-1)(y-2), y(0) = 1,2$
4. $\frac{dy}{dt} = y(y-1)(y-2), y(0) = -1$
5. $\frac{dy}{dt} = e^y - 1, y(0) = 1$

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Fourth Meeting : I. Introduction to Differential Equations
- Differential Equations Solution

As mentioned before, one of our goals in this course is to solve or to find the solutions of differential equations.

Definition Solution of a Differential Equation

Any function f defined on some interval I, which when substituted into a differential equation reduces equation to an identity, is said to be a solution of the equation on the interval. In other words, f is said to be a solution if it possess at least *n* derivatives and satisfies the differential equation, $y = f(x)$.

Consider the following differential equation

$$\frac{dy}{dx} = -\frac{x}{y}$$

Explicit solution

A solution in which the dependent variable is expressed in terms of the independent variable and constant is said to be an explicit solution. Specifically, an explicit solution of a differential equation that is identically zero on an interval I is said to be a trivial solution.

Example : $y = \pm\sqrt{4 - x^2}$ is an explicit solution of differential equation above.

Implicit solution

A relation $G(x, y) = 0$ is said to be an implicit solution of an ordinary differential equation on an interval I provided there exists at least one function $\emptyset$ that satisfies the relation as well as the differential equation on I. in other words, $G(x, y) = 0$ defines the function $\emptyset$ implicitly.

Example : $x^2 + y^2 - 4 = 0$ is an implicit solution of differential equation above.

General solution

A solution containing single or multiple arbitrary constant is called general solution.

Example : $x^2 + y^2 - c = 0$ for every constan $c \geq 0$ is general solution of differential equation above.

Particular solution

A solution of a differential equation that is free of arbitrary parameters is called a particular solution.

Example : suppose $y(0) = 4$ then $c = 16$, therefore $x^2 + y^2 - 16 = 0$ is particular solution of differential equation where $y(0) = 4$.

Exercises 4.

1. Show that $y = 4e^{2x} + 2e^{-3x}$ is a solution of differential equation as follows

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

2. Show that $y = (x^2 + c)e^{-x}$ is the solution of $\frac{dy}{dx} + y = 2xe^{-x}$. Given that $y(0) = 2$, find the particular solution.

3. Show that $y = c_1 e^{4x} + c_2 e^{-3x}$ is the solution of $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0$ and then find the particular solution if it's known that $y(0) = 5$ and $y^{'}(0) = 6$.
4. Show that for some choice of the arbitrary constants c_1 and c_2, the form $y = c_1 \sin x + c_2 \cos x$ is the solution of $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$ but the boundary problems $y(0) = 0$ and $y(\pi) = 1$ doesn't possess the solution.
5. Find the value of c_1, c_2 and c_3 such that $y = c_1 x + c_2 x^2 + c_3 x^3$ is the solution of

$$x^3 \frac{d^3y}{dx^3} - 3x^2 \frac{d^2y}{dx^2} + 6x \frac{dy}{dx} - 6y = 0$$

That satisfying three conditions

$$y(2) = 0, \quad y^{'}(2) = 2, \quad y"(2) = 6$$

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Fifth Meeting : II. First Order Differential Equations for Which Exact Solutions are obtainable
- Standard forms of First Order Differential Equations
- Exact Equation

In this chapter, we will discuss the solution of first order differential equation. The main idea is to check whether it is an exact or nonexact. For Nonexact first order differential equation, we should use integrating factor to transform nonexact differential equation into the new differential equation which is exact.

II.A Standard forms of First Order Differential Equations

First order differential equation can be expressed as derivative form

$$\frac{dy}{dx} = f(x, y) \dots \dots (2.1)$$

or as differential form

$$M(x, y)\, dx + N(x, y)\, dy = 0 \dots \dots (2.2)$$

Definition 2.1

Let F be a function of two real variables such that F has continuous first partial derivatives in a domain D. The total differential dF of the function F is defined by the formula

$$dF(x, y) = \frac{\partial F(x, y)}{\partial x}\, dx + \frac{\partial F(x, y)}{\partial y}\, dy$$

For all $(x, y) \in D$.

Definition 2.2

In a domain D, if there exists a function F of two real variables such that expression

$$M(x,y)dx + N(x,y)dy$$

equals to the total differential $dF(x,y)$ for all $(x,y) \in D$ then this expression is called an exact differential.

In other words, the expression above is an exact differential in D if there exists a function F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \text{ and } \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

for all $(x,y) \in D$.

EXACT EQUATION

If $M(x,y)dx + N(x,y)dy$ is an exact differential, then the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is called **an exact differential equation.**

The following theorem is used to identify whether a differential equation is an exact or not.

Theorem

Consider the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Where M and N have continuous first partial derivatives at all points (x,y) in a rectangular domain D.

1. If the differential equation is exact in D, then

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

For all $(x,y) \in D$

2. Conversely, if

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

For all $(x,y) \in D$, then the differential equation is exact in D.

Example 2.1

Determine the following differential equation is exact or nonexact.

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0$$

Answer :

The differential equation is expressed in derivative form. Transformed into differential form, we get

$$(2xy - 9x^2)\, dx\ (2y + x^2 + 1)\, dy = 0$$

Suppose $M(x, y) = 2xy - 9x^2$ and $N(x, y) = 2y + x^2 + 1$, then the differential equation above is exact if and only if

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}.$$

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial(2xy - 9x^2)}{\partial y} = 2x$$

While

$$\frac{\partial N(x,y)}{\partial x} = \frac{\partial x\ (2y + x^2 + 1)}{\partial x} = 2x$$

Because $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$, then the differential equation above is an exact.

Exercises 5.

Determine whether it is an exact differential equation or not. Explained!

1. $[1 + \ln(xy)]\, dx + \frac{x}{y}\, dy = 0$
2. $x^2y\, dx - (xy^2 + y^3)\, dy = 0$
3. $(y + 3x^2)\, dx + x\, dy = 0$
4. $[\cos(xy) - xy\sin(xy)]\, dx - x^2\sin(xy)\ dy = 0$
5. $ye^{xy}\, dx + (2y - xe^{xy})\, dy = 0$

More exercises can be found in the references.

By using this theorem, the solution for exact differential equation above is a function F such that satisfies

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \text{ and } \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

for all $(x, y) \in D$.

Therefore,

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{\partial F(x,y)}{\partial x}\,dx + \frac{\partial F(x,y)}{\partial y}\,dy$$

$$dF(x,y) = 0$$

If we manipulate it for several operation, we obtain the solution can be expressed into

$$F(x,y) = \int M(x,y)\ \partial x + \int \left[N(x,y) - \int \frac{\partial M(x,y)}{\partial y}\ \partial x \right] dy$$

Example 2.2

Solve the differential equation below

$$(3x + 2y)dx + (2x + y)dy = 0$$

Answer :

First we have to figure it out whether it's an exact differential equation or not.

Suppose $M(x,y) = 3x + 2y$ and $N(x,y) = 2x + y$

Therefore, we get

$$\frac{\partial M(x,y)}{\partial y} = 2 \quad and \quad \frac{\partial N(x,y)}{\partial x} = 2$$

Because $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$, then the differential equation above is an exact differential equation.

So, there exist a function F such that satisfies

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \text{ and } \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

We have

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) = 3x + 2y$$

Then

$$F(x,y) = \int (3x + 2y)\ \partial x + \theta(y) = \frac{3}{2}x^2 + 2y + \theta(y)$$

Derived it for y,

$$\frac{\partial F(x,y)}{\partial y} = 2 + \theta^{'}(y)$$

Because $\frac{\partial F(x,y)}{\partial y} = N(x,y)$, then

$$2 + \theta^{'}(y) = 2 \rightarrow \theta^{'}(y) = 0$$

Here we get,

$$\theta(y) = \int \theta^{'}(y)\, dy = \int 0\ dy = c$$

From the operations above, we obtain the solution for the differential equation is

$$F(x,y) = \frac{3}{2}x^2 + 2y + c$$

Exercise 6.

1. Solve the exact differential equations in Exersice 5.
2. Solve the following differential equations.
   a. $(2xy)\,dx + (x^2 + 1)\,dy = 0$
   b. $(y^2 + \cos x)\,dx + (2xy + \sin y)\,dy = 0$
   c. $(4e^{2x} + 2xy - y^2)\,dx + (x - y)^2\,dy = 0$

Seventh Meeting : II. First Order Differential Equations for Which Exact Solutions are obtainable
- Method of Grouping
- Integrating Factor

Another way to find the solution of differential equation is by method of grouping that is grouping the terms of differential equation in such a way that its left member appears as the sum of certain exact differentials.

Example

$$(2x\cos y + 3x^2 y)\, dx + (x^3 - x^2 \sin y - y) dy = 0$$

Answer :

We can rewrite the differential equation above into the following term

$$(2x\cos y\ \ dx - x^2 \sin y\ \ dy) + (3x^2 y\, dx + x^3 dy) - y dy = 0$$

By using total difference for each term under the blankets, here we get,

$$d(x^2\cos y) + d(x^3 y) - d\left(\frac{1}{2}y^2\right) = 0$$

Integrating both sides, we have

$$x^2\cos y + x^3 y - \frac{1}{2}y^2 = 0$$

So, the solution of the differential above is $x^2\cos y + x^3 y - \frac{1}{2}y^2 = 0$.

Exercise 7.1

Solve the following equations by method of grouping.

1. $(3x + 2y)\, dx + (2x + y) dy = 0$
2. $(y^2 + 3)\, dx + (2xy - 4) dy = 0$
3. $(6xy + 2y^2 - 5) dx - (x^3 + y)\, dy = 0$

4. $(a^2 + 1)\cos r\ dr + 2a\sin r\ \ da = 0$
5. $(y\sec^2 x + \sec x\tan x)\ dx + (\tan x + 2y)dy = 0$

From the previous meeting, it has been discussed differential equation which is exact. How if the equation isn't exact? Here, we introduced integrating factor that is a factor when multiplied into the equation, it will transform the equation into exact.

Definition

If the differential equation

$$M(x,y)\ \ dx + N(x,y)\ \ dy = 0$$

is not exact in a domain D but the differential equation

$$\mu(x,y)M(x,y)\ dx + \mu(x,y)N(x,y)\ dy = 0$$

is exact in D, then $\mu(x,y)$ is called an integrating factor of the differential equation.

Note : Solution of the first equation is equals with the second equation.

Exercises 7.2

1. Consider the differential equation

$$(4x + 3y^2)\ dx + 2xy\ dy = 0$$

   a. Show that this equation is not exact
   b. Find the integratiiing factor of the form x^n, where n is a positive integer.
   c. Multiply the given equation through by the integrating factor found in (b) and solve the resulting exact equation.

2. Consider the differential equation

$$(y^2 + 2xy)dx - x62\ dy = 0$$

   a. Show that this equation is not exact.
   b. Multiply the given equatttion through by y^n, where n is an integer and then determine n so that y^n is an integrating factor of the given equation.

c. Multiply the given equation through by the integrating factor found in (b) and solve the resulting exact equation.
d. Show that $y = 0$ is a solution of the original nonexact equation but is not a solution of the essentially equivalent exact equation found in step (c).
e. Graph several integral curves of the original equation, including all those whose equations are (or can be writen) in some special form.

Eighth Meeting : II. First Order Differential Equations for Which Exact Solutions are obtainable
- Separable Differential Equations

Next, we will discuss about some kind of nonexact differential equation and how it is solved.

A. SEPARABLE EQUATIONS

Consider to the general form of differential equation in differential form. The differential equation is called **separable differential equations** if f or M and N can be expressed into multiplication of function of each variable. In other words, M and N can be expressed into formula in the definition as follows :

Definition

An equation of the form

$$F(x)G(y)dx + f(x)g(y)dy = 0$$

or

$$\frac{dy}{dx} = g(x)h(y)$$

is called an equation with variables separable or simply a separable equation.

As we can see that separable equation which is expressed in differential form, in general, is not exact, but it possesses an obvious integrating factor, namely $\frac{1}{f(x)G(y)}$ where $f(x_1) \neq 0$ and $G(y_1) \neq 0$ for some x_1 and y_1. Therefore, if we multiply this equation by this expression, we separate the variables, reducing it to the essentially equivalent equation

$$\frac{F(x)}{f(x)}\,dx + \frac{g(y)}{G(y)}\,dy = 0$$

This last equation is exact since

$$\frac{\partial}{\partial y}\left(\frac{F(x)}{f(x)}\right) = 0 = \frac{\partial}{\partial x}\left(\frac{g(y)}{G(y)}\right)$$

The solution of the separable equation then simply can be resulted by integrating each variable

$$\int \frac{F(x)}{f(x)}\,dx = \int \frac{g(y)}{G(y)}\,dy$$

If there exists x_1 and or y_1 such that $f(x_1) = G(y_1) = 0$, then we must determine whether any of these solutions of the original equation which were lost in the formal separation process.

Example.

Solve

$$xy^4\,dx + (y^2+2)e^{-3x}\,dy = 0$$

Answer :

This equation is an exact differential equation. We define the integrating factor of its equation is

$$\frac{1}{y^4 e^{-3x}}, \quad y^4 \neq 0.$$

Here, we obtain

$$\frac{x}{e^{-3x}}\,dx + \frac{(y^2+2)}{y^4}\,dy = 0$$

$$\frac{x}{e^{-3x}}\,dx = -\frac{(y^2+2)}{y^4}\,dy$$

Using integration by parts

$$\int \frac{x}{e^{-3x}}\,dx = \int -\frac{(y^2+2)}{y^4}\,dy$$

yields

$$-\frac{1}{x}+\frac{2}{x^2}+\frac{1}{y}-\frac{1}{y^3}=c$$

Then, we have to check whether $y=0$ is a solution from its origin equation which is loss in separation process or not.

$$xy^4\,dx+(y^2+2)e^{-3x}\,dy=0\rightarrow\frac{dy}{dx}=-\frac{xy^4}{(y^2+2)e^{-3x}}$$

Here, we observe that $y=0$ is also a solution.

Exercise 8.

Solve the differential equations below

1. $(4y+yx62\,)dy-(2x+xy^2)dx=0$
2. $(1+x62+y^2+x^2y^2)\,dy=y^2dx$
3. $2y(x+1)\,dy=xdx$
4. $x^2y^2dy=(y+1)\,dx$
5. $e^y\sin 2x\,dx+\cos x\,(e^{2y}-y)dy=0$

Ninth Meeting : II. First Order Differential Equations for Which Exact Solutions are obtainable
- Homogeneous Differential Equations

B. HOMOGENEOUS EQUATIONS

Definition

A function f of two variables *x* and *y* is said homogeneous if it satisfies

$$f(tx, ty) = f(x, y)$$

for any number *t*.

Example

$$f(x, y) = \frac{x^2 + y^2}{xy}$$

is a homogeneous function because for every number t,

$$f(tx, ty) = \frac{(tx)^2 + (ty)^2}{(tx)(ty)} = \frac{t^2(x^2 + y^2)}{t^2 xy} = \frac{x^2 + y^2}{xy} = f(x, y)$$

Definition

A function M of two variables x and y is said to be homogenous of degree-n if it satisfies

$$M(tx, ty) = t^n M(x, y)$$

Example

$$M(x, y) = x^3 + x^2 y$$

is a homogenous of degree-3 function because for every number t,

$$M(tx, ty) = (tx)^3 + (tx)^2(ty) = t^3x^3 + t^3x^2y = t^3(x^3 + x^2y) = t^3 M(x, y)$$

By here, we can define homogeneous differential equation as follows

Definition

A first differential equation

$$M(x,y)dx + N(x,y)dy = 0 \text{ or } \frac{dy}{dx} = f(x,y)$$

is called **homogeneous differential equation** if *M* and *N* are homogeneous functions of the same degree *n* or *f* is a homogenous function.

To solve homogeneous differential equation, we can use the following theorem:

Theorem

If

$$M(x,y)dx + N(x,y)dy = 0$$

is a homogeneous differential equation, then the change of variables $y = vx$ transforms it into a separable differential equation in the variables v and x.

Here, we can summarize the steps of solving homogeneous differential equation as follow:

1. Recognize that the equation is a homogeneous differential equation
2. Transforms it into a separable differential equation by $y = vx$
3. Solve the separable differential equation in the variable v and x by integrating by parts.
4. Inverse it into the origin differential equation $v = \frac{y}{x}$.

Example.

Solve this differential equation

$$2xy\, dx + (x^2 + 1)dy = 0$$

Answer:

As we can see that M and N are homogeneous functions with the same degree 2, therefore the we should transform it into separable equation by substituting $y = vx$. Here we get, $dy = vdx + xdv$.

Substitute into the equation, we get

$$2x(vx)\,dx + (x^2+1)(v\,dx + x\,dv) = 0$$

$$2x^2v\,dx + (x^2+1)v\,dx + (x^3+x)\,dv = 0$$

$$(3x^2+1)v\,dx + (x^3+x)\,dv = 0$$

The last equation is separable equation. To solve this equation, we multiply it with integrating factor $\frac{1}{v(x^3+x)}$. We get,

$$\frac{(3x^2+1)}{x^3+x}\,dx + \frac{1}{v}\,dv = 0$$

By integrating it in both side, we get

$$\int \frac{(3x^2+1)}{x^3+x}\,dx + \int \frac{1}{v}\,dv = \ln|c|$$

$$\ln|x^3+x| + \ln|v| = \ln|c|$$

$$\ln|x^3+x|\,|v| = \ln|c|$$

$$(\,x^3+x)v = c \rightarrow \frac{(x^3+x)y}{x} = c \rightarrow x^2y + y = c$$

So, the solution of the differential equation above is $x^2y + y = c$.

Exercise 9.

Solve the following differential equations.

1. $(x+2y)dx + (2x-y)dy = 0$
2. $(3x-y)dx - (x+y)dy = 0$
3. $(2xy+3y^2)\,dx - (2xy+x^2)\,dy = 0$
4. $y^3dx + (x^3 - xy^2)\,dy = 0$
5. $(2s^2+2st+t^2)\,ds + (s^2+2st-t^2)\,dt = 0$

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Tenth Meeting : II. First Order Differential Equations for Which Exact Solutions are obtainable
- Linear Differential Equations

C. LINEAR EQUATIONS

Definition

A first-order ordinary differential equation is linear in the dependent variable y and the independent variable x if it is, or can be, written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

By several multiplications, equation above can also be expressed as

$$\{P(x)y - Q(x)\}\,dx + dy = 0$$

Suppose the integrating factor

$$\mu(x) = e^{\int P(x)\,dx}$$

The general solution of linear differential equation as follows

$$y = \mu^{-1}(x)\left[\int \mu(x)\,Q(x)\,dx + c\right]$$

Steps to solve linear differential equation are

1. If the differential equation is given as

$$a(x)\frac{dy}{dx} + b(x)y = c(x)$$

Rewrite it in the form

$$\frac{dy}{dx} + p(x)y = q(x)$$

2. Find the integrating factor

$$\mu(x) = e^{\int P(x)\,dx}$$

3. Evaluate the integral

$$\int \mu(x)\,Q(x)\,dx$$

4. Write down the general solution

$$y = \mu^{-1}(x)\left[\int \mu(x)\, Q(x)\ dx + c\right]$$

5. If it is given an initial value, use it to find the constant C.

Example. Solve

$$\frac{dy}{dx} - 3y = 0$$

Answer :

1. As we can see from the formula above that $P(x) = -3$ and $Q(x) = 0$. Therefore, the integrating factor of the equation is

$$\mu(x) = e^{\int -3\, dx} = e^{-3x}$$

2. Multiply both sides with the integrating factor, yields

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 0$$

3. Evaluate

$$\int \mu(x)\, Q(x)\ dx = \int e^{-3x}.0\ dx = c_1$$

4. The general solution is

$$y = \mu^{-1}(x)\left[\int \mu(x)\, Q(x)\ dx + c\right]$$
$$= e^{3x}[c_1 + c]$$
$$y = c_0 e^{3x}$$

Exercise 10.

1. $\frac{dy}{dx} + \frac{3y}{x} = 6x^2$
2. $\frac{dy}{dx} + 3y = 3x^2e^{-x}$
3. $\frac{dy}{dx} + 4xy = 8x$
4. $x\frac{dy}{dx} + (3x + 1)y = e^{-3x}$
5. $(x + 1)\frac{dy}{dx} + (x + 2)y = 2xe^{-x}$

D. BERNOULLI EQUATIONS

Definition

An equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

is called a **Bernoulli differential equation.**

To solve Bernoulli differential equation, where n ≠ 0 or n ≠ 1, we can use the following theorem.

Theorem

Suppose n ≠ 0 or n ≠ 1, then the transformation $v = y^{1-n}$ reduces the Bernoulli equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

into a linear equation in v.

Therefore, the steps to solve Bernoulli Differential Equation are :

1. Recognize that the differential equation is a Bernoulli Differential Equation. Then find the parameter n from the equation.
2. The main idea is to transform the Bernoulli Differential Equation into Linear Differential Equation, therefore the things that we should do are
   a. Divided both side with y^n.
   b. Transform the equation by $v = y^{1-n}$. Here we get, $\frac{dv}{dy} = (1-n)y^{-n}$.

c. By using manipulations, we ge

$$\frac{dv}{dx} + P^*(x)v = Q^*(x)$$

d. Solve the linear equation.

Example

Solve the differential equation below

$$x\frac{dy}{dx} + y = x^2y^2$$

Answer :

To solve the differential equations above, first we have to divided both side with xy^2. Here we get,

$$y^{-2}\frac{dy}{dx} + x^{-1}y^{-1} = x$$

Suppose $v = y^{-1}$ then $\frac{dv}{dy} = -y^{-2}$ or else $\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$. By here, we get

$$\frac{dv}{dx} + x^{-1}v = x$$

The differential equation above is linear equation. To solve the equation, we use integrating factor

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln|x|} = x$$

Then the solution is

$$v = \mu^{-1}(x)\left[\int \mu(x)\, Q(x)\ dx + c\right]$$

$$v = x^{-1}\left[\int x^2\ dx + c\right] \to v = x^{-1}\left[\frac{1}{3}x^3 + c\right] = \frac{1}{3}x^2 + cx^{-1}$$

$$y^{-1} = \frac{1}{3}x^2 + cx^{-1}$$

$$1 = \frac{1}{3}x^2y + cx^{-1}y$$

Exercise 11.

Solve the following differential equations.

1. $\frac{dy}{dx} - \frac{1}{x}y = xy^2$
2. $\frac{dy}{dx} + \frac{y}{x} = y^2$
3. $\frac{dy}{dx} + \frac{1}{3}y = e^x y^4$
4. $x\frac{dy}{dx} + y = xy^3$
5. $\frac{dy}{dx} + \frac{2}{x}y = -y^2x^2\cos x$

Thirteenth Meeting : II. First Order Differential Equations for Which Exact Solutions are obtainable
- Transformation

E. SPECIAL INTEGRATING FACTOR

For the last four meetings, we had solved four different differential equations which are not exact by transforming it into new equation which are exact. One of transformation that can be done is using integrating factor. Integrating factor can be used to transform non exact differential equation into exact differential equation. Unfortunately, there is no general expression for transformation. Nevertheless, we shall consider a few of transformation for special differential equations.

Consider the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

Case	Integrating factor
$\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]$ Depends upon *x* only	$e^{\int \frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] dx}$
$\frac{1}{M(x,y)}\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right]$ Depends upon y only, then	$e^{\int \frac{1}{M(x,y)}\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right] dy}$

Example

Solve it.

$$(5xy + 4y^2 + 1)\, dx + (x^2 + 2xy)\, dy = 0$$

Answer :

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial(5xy + 4y^2 + 1)}{\partial y} = 5x + 8y$$

$$\frac{\partial N(x,y)}{\partial x} = \frac{\partial(x^2 + 2xy)}{\partial x} = 2x + 2y$$

$$\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] = \frac{1}{x^2 + 2xy}[5x + 8y - (2x + 2y)]$$

$$= \frac{1}{x^2 + 2xy}[3x + 6y]$$

$$= \frac{3}{x}$$

Because $\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]$ depends on x only, then the integrating factor of equation above is

$$e^{\int \frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] dx} = e^{\int \frac{3}{x} dx} = e^{3 \ln x} = x^3$$

Multiply both equation with x^3, we get

$$x^3(5xy + 4y^2 + 1)\, dx + x^3(x^2 + 2xy)\, dy = 0$$

$$(5x^4y + 4x^3y^2 + 1)\, dx + (x^5 + 2x^4y)\, dy = 0$$

By using method of grouping as follow

$$[5x^4y\, dx + x^5 dy] + [4x^3y^2\, dx + 2x^4y\, dy] + 1\, dx = 0$$

and then define the total derivative as

$$d(x^5y) + d(x^4y^2) + dx = 0$$

Integrating it, we get the solution as follows

$$x^5y + x^4y^2 + x = c.$$

F. SPECIAL TRANSFORMATIONS

Consider the differential equation

$$(a_1x + b_1y + c_1)\, dx + (a_2x + b_2y + c_2)\, dy = 0$$

where $a_1, b_1, c_1, a_2, b_2,$ and c_2 are constants

Case	Special Transformation
$\frac{a_2}{a_1} \neq \frac{b_2}{b_1}$	$x = X + h, y = Y + k$ Where (h,k) is the solution of the system $a_1h + b_1k + c_1 = 0$

	$a_2h + b_2k + c_2 = 0$ It will reduce into the homogenous differential equation $(a_1X + b_1Y)\ dX + (a_2X + b_2Y)\ dY = 0$
$\frac{a_2}{a_1} = \frac{b_2}{b_1} = k$	$z = a_1x + b_1y$ will reduce the equation into separable differential equation

Example

Solve this differential equation.

$$(3x - y + 1)\ dx - (6x - 2y - 3)dy = 0$$

Answer :

We can rewrite the equation above with

$$(3x - y + 1)\ dx + (-6x + 2y + 3)dy = 0$$

For the equation above, $a_1 = 3, b_1 = -1, a_2 = -6$ and $b_2 = 2$. So, we get

$$\frac{a_2}{a_1} = \frac{b_2}{b_1} = -2$$

Therefore, we can solve the equation by reduced it into separable differential equation using $z = 3x - y$. This is implied that $dz = 3dx - dy$ or $dy = 3dx - dz$.

Substitute it into the question, we get

$$(z + 1)\ dx + (-2z + 3)(3dx - dz) = 0$$

$$(z + 1 - 6z + 9)\ dx + (2z - 3)dz = 0$$

$$(-5z + 10)dx + (2z - 3)dz = 0$$

$$dx - \frac{2z - 3}{5z - 10}\ dz = 0$$

By using some manipulations we get solution

$$x - \frac{2}{5}\left(z + \frac{1}{2}\ln|10z - 20|\right) = c$$

atau

$$x - \frac{2}{5}\left(3x - y + \frac{1}{2}\ln|10(3x - y.) - 20|\right) = c$$

Exercise 15.

Find the integrating factor and then solve the differential equations below.

1. $(4xy + 3y^2 - x)\ dx + x(x + 2y)\ dy = 0$
2. $y(x + y)\ dx + (x + 2y - 1)dy = 0$
3. $y(x + y + 1)\ \ dx + x(x + 3y + 2)\ dy = 0$
4. $(6x + 4y + 1)\ dx + (4x + 2y + 2)dy = 0$
5. $(3x - y - 6)dx + (x + y + 2)dy = 0$
6. $(2x + 3y + 1)dx + (4x + 6y + 1)dy = 0$
7. $(4x + 3y + 1)dx + (x + y + 1)dy = 0$

Sixteenth Meeting : III. Applications of First Order Equations

- Orthogonal Trajectories

As we had learned at the first meeting that differential equations were behind many mathematical models. In this chapter, we will discuss about the application of differential particularly for first order equations.

Before we study about the application of first order differential equation, first of all we will study about orthogonal trajectories and oblique trajectories.

Suppose it is given differential equation in derivative form

$$\frac{dy}{dx} = f(x, y)$$

The solution of its differential equation

$$F(x, y) = c \text{ or } F(x, y, c) = 0$$

When we draw $F(x, y, c) = 0$, we can get a lot of curve depends on the value of c. By here, set of curves of $F(x, y, c) = 0$ is called one-parameter family of curves.

Definition

Let

$$F(x, y, c) = 0$$

Be a given one-parameter family of curves in the XY plane. A curve that intersects the curves of the family at right angles is called an orthogonal trajectory of the given family.

Consider two families of curves F_1 and F_2. We say that F_1 and F_2 are orthogonal whenever any curve from F_1 intersects any curve from F_2, the two curves are orthogonal at the point of intersection.

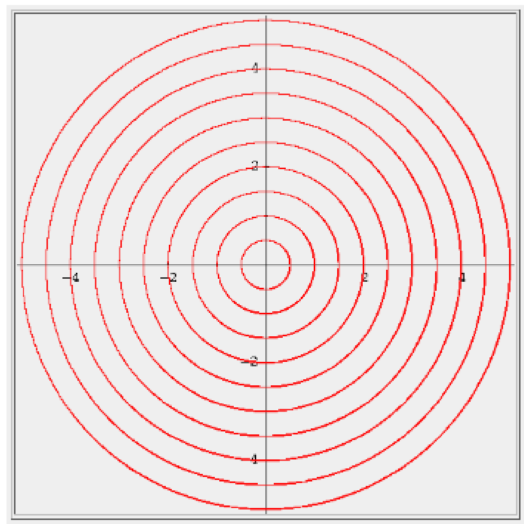
Procedures to find the ortogonal trajectories of the family of curve, $F(x,y,c)=0$

1. Find the differential equation of the given family,
$$\frac{dy}{dx}=f(x,y)$$
2. Find the differential equation of the orthogonal trajectories by replacing $f(x,y)$ by its negative reciprocal
$$-\frac{1}{f(x,y)}$$
3. Solve the new differential equation.
$$\frac{dy}{dx}=-\frac{1}{f(x,y)}$$

Example

Find the orthogonal trajectories of the family of curve as follow :

$$x^2+y^2=C$$



Answer :

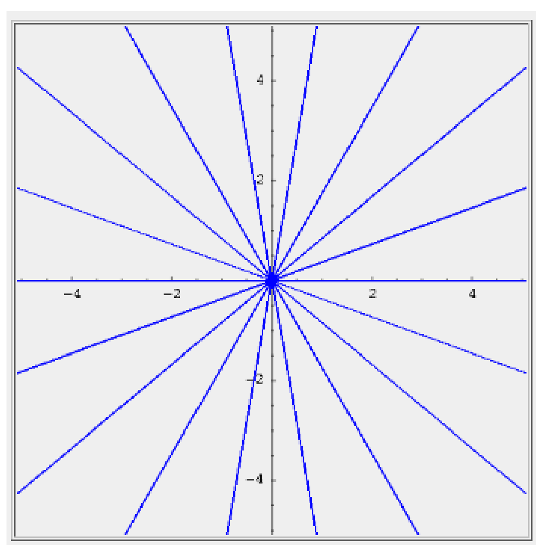
The differential equation related to the family of curve above is

$$2xdx+2ydy=0$$

or

$$\frac{dy}{dx}=-\frac{x}{y}$$

Therefore we get the differential equation of the orthogonal trajectories



$$\frac{dy}{dx}=\frac{y}{x}$$

$$\frac{1}{x}dx-\frac{1}{y}dy=0$$

Integrating both sides we get

$$\ln|x|-\ln|y|=\ln c_0$$

$$\frac{x}{y}=c$$

Exercise 16.

1. $y = Ce^{-2x}$
2. $x^2 - y^2 = C$
3. $y = Cx^2$
4. $x^2 + (y - c)^2 = C^2$

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Seventeenth Meeting : III. Applications of First Order Equations
- Oblique Trajectories

Definition

Let

$$F(x, y, c) = 0$$

be a one-parameter family of curves. A curve that intersects the curves of this family at a constant angle $\alpha \neq 90^0$ is called an oblique trajectory of the given family.

Procedures to find the oblique trajectories of the family of curve, $F(x, y, c) = 0$ at angle $\alpha \neq 90^0$

1. Find the differential equation of the given family,
$$\frac{dy}{dx} = f(x, y)$$
2. Find the differential equation of the orthogonal trajectories by replacing $f(x, y)$ by expression
$$\frac{f(x, y) + \tan\alpha}{1 - f(x, y)\tan\alpha}$$
3. Solve the new differential equation.
$$\frac{dy}{dx} = \frac{f(x, y) + \tan\alpha}{1 - f(x, y)\tan\alpha}$$

Example

Find a family of oblique trajectories that intersect the family of straight lines $y = cx$ at angle 45^0.

Answer :

The derivative of the straight lines above is

$$\frac{dy}{dx} = c$$

Therefore we get, the derivate of the oblique trajectory is

$$\frac{dy}{dx} = \frac{c + \tan 45^0}{1 - c \tan 45^0} = \frac{c + 1}{1 - c}$$

$$y = \frac{c + 1}{1 - c} x$$

$$\text{or } y = dx, \text{where } d = \frac{c + 1}{1 - c}$$

Exercise 17.

1. Find a family of oblique trajectories that intersect the family of circles $x^2 + y^2 = c$ at angle 45^0.
2. Find a family of oblique trajectories that intersect the family of parabolas $y^2 = cx$ at angle 60^0.
3. Find a family of oblique trajectories that intersect the family of curves $x + y = cx^2$ at angle α such that $\tan \alpha = 2$.

18th,19th Meeting : III. Applications of First Order Equations
- Application in Mechanics Problems and Rate Problems

Next, it will be presented examples where differential equations are widely applied to model natural phenomena, engineering systems and many other situations.

III.A Exponential Growth – Population

Let $P(t)$ be a quantity that increases with time t. Suppose the rate of increase is proportional to the same quantity at that time. At first, the quantity is $P_0 > 0$. This problem can be modeled into differential equation as follows

$$\frac{dP}{dt} = kP, k > 0$$

where $P(0) = P_0$.

This equation is called an **exponential growth model.**

The solution of its equation is

$$P(t) = P_0 e^{kt}$$

III.B Exponential Decay

Let $M(t)$ be the amount of a product that decreases with time t. Suppose the rate of decrease is proportional to the amount of product at any time t. Suppose that at initial time $t = 0$, the amount is M_0. This problem can be modeled into differential equation as follows

$$\frac{dM}{dt} = -kM, k > 0$$

Where $M(0) = M_0$.

This equation is called an **exponential decay model**.

The solution of its equation is

$$M(t) = M_0 e^{-kt}$$

III.C Falling object

An object is dropped from a height at time $t = 0$. If $h(t)$ is the height of the object at time t, $a(t)$ is the acceleration and $v(t)$ is the velocity, then the relationship between a, v and h are as follows :

$$a = \frac{dv}{dt}, \qquad v = \frac{dh}{dt}$$

For a falling object, $a(t)$ is constant and is equal to $g = -9.8m/s$.

Suppose that h_0 and v_0 are the initial height and initial velocity, respectively. The solution of the equations is

$$h(t) = \frac{1}{2}gt + v_0 t + h_0$$

and is called the height of falling object at time t.

III.D Newton's Law of Cooling

It is a model that describes, mathematically, the change in temperature of an object in a given environment. The law states that the rate of change (in time) of the temperature is proportional to the difference between the temperature T of the object and the temperature T_e of the environment surrounding the object.

Assume that at t = 0, the temperature is T_0.

This problem can be modeled as differential equation as follows

$$\frac{dT}{dt} = -k(T - T_e)$$

Where $T(0) = T_0$.

The final expression for $T(t)$ is given by

$$T(t) = T_e + (T_0 - T_e)e^{-kt}$$

This last expression shows how the temperature T of the object changes with time.

III.E RL Circuit

Let us consider the RL (resistor R and inductor L) circuit shown below. At $t = 0$ the switch is closed and current passes through the circuit. Electricity laws state that the voltage across a resistor of resistance R is equal to R i and the voltage across an inductor L is given by $L\frac{di}{dt}$ (i is th current). Another law gives an equation relating all voltages in the above circuit as follows :

$$L\frac{di}{dt} + iR = E$$

Where E is a constant voltage and $i(0) = 0$.

Rewrite the differential equation above as

$$\frac{L}{E - iR}\frac{di}{dt} = 1$$

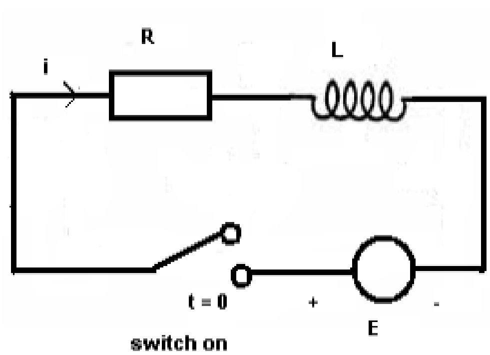
$$-\frac{L}{R}\frac{-R}{E - iR}di = dt$$

Integrate both sides, we get

$$-\frac{L}{R}\ln|E - iR| = t + c$$

By using the initial condition, the formula of i is

$$i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$



Exercise 18, 19.

Find another application of first order linear differential equation.

22nd, 23rd Meeting : IV. Explicit Methods of Solving Higher Order Linear Differential Equations
- Basic Theory of Linear Differential Equations

The most general n^{th} order linear differential equation is

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_o(x)y = G(x)$$

Where you'll hopefully recall that

$$y^{(m)} = \frac{d^m y}{dx^m}$$

It's called homogeneous equation if $G(x) = 0$ and is called nonhomogeneous equation if $G(x) \neq 0$.

Superposition Principle

Let $y_1, y_2, \dots, y_m$ be solutions of the homogeneous equation above. Then, the function

$$y = c_1y_1 + c_2y_2 + c_3y_3 + \cdots + c_ny_n$$

is also solution of the equation. This solution is called a linear combination of the functions $\{y_i\}$.

The general solution of the homogeneous equation is given by

$$y = c_1y_1 + c_2y_2 + c_3y_3 + \cdots + c_ny_n$$

Where $(c_1, c_2, c_3, \dots, c_n)$ are arbitrary constants and $\{y_1, y_2, y_3, \dots, y_n\}$ are n-solutions of the homogeneous equation such that

$$W(y_1, y_2, y_3, \dots, y_n)(x) = \begin{vmatrix} y_1(x) & y_2(x) & y_3(x) & \cdots & y_n(x) \\ y'_1(x) & y'_2(x) & y'_3(x) & & y'_n(x) \\ \vdots & & \vdots & & \vdots \\ y^{(n-1)}{}_1(x) & y^{(n-1)}{}_2(x) & y^{(n-1)}{}_3(x) & \cdots & y^{(n-1)}{}_n(x) \end{vmatrix} \neq 0$$

In this case, we will say that $\{y_1, y_2, y_3, \dots, y_n\}$ are linearly independent. The function $W(y_1, y_2, y_3, \dots, y_n) = W$ is called the Wronskian of $\{y_1, y_2, y_3, \dots, y_n\}$.

The general solution of the nonhomogeneous equation is given by

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots + c_n y_n + y_p = y_c + y_p$$

Where $(c_1, c_2, c_3, \dots, c_n)$ are arbitrary constants and $\{y_1, y_2, y_3, \dots, y_n\}$ are linearly independent solutions of the associated homogeneous equation, and y_p is a particular solution of nonhomogeneous equation.

Example.

Given that e^x, e^{-x} and e^{2x} are the solutions of

$$y^{(3)} - 2y" - y' + 2y = 0$$

Show that these solutions are linearly independent on every real interval.

Answer :

$$W(e^x, e^{-x}, e^{2x}) = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} = e^{2x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = -6e^{2x} \neq 0$$

Because the wronskrian is not equals to zero, then these solutions are linearly independent.

Exercise 22

1. Consider the differential equation

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

   a. Show that x and x^2 are linearly independent solutions of this equation on the interval $0 < x < \infty$
   b. Write the general solution of the given equation.
   c. Find the solution that satisfies the conditions $y(1) = 3, y^{'}(1) = 2$. Explain why this solution is unique. Over what interval is this solution defined?

2. Consider the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$$

   a. Show that x^2 and x^{-2} are linearly independent solutions of this equation on the interval $0 < x < \infty$.
   b. Write the general solution of the given equation.
   c. Find the solution that satisfies the condition $y(2) = 3, y'(2) = -1$. Explain why this solution is unique. Over what interval is this solution defined?

3. Consider the differential equation

$$y" - 5y' + 4y = 0$$

   a. Show that each of the functions e^x, e^{4x} and $2e^x - 3e^{4x}$ is a solution of this equation on the interval $-\infty < x < \infty$.
   b. Show that e^x and e^{4x} are linearly independent on $-\infty < x < \infty$.
   c. Show that the solution e^x and $2e^x - 3e^{4x}$ are also linearly independent on $-\infty < x < \infty$.

24th, 25th, 26th Meeting : IV. Explicit Methods of Solving Higher Order Linear Differential Equations

- The Homogeneous Linear Equation with Constant Coefficients

In this chapter we will be looking exclusively at linear second order differential equations. The most general linear second order differential equation is in the form

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_o(x)y = G(x)$$

In fact, we will rarely look at non-constant coefficient linear second order differential equations. In the case where we assume constant coefficients we will use the following differential equation.

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_o y = G(x)$$

Initially we will make our life easier by looking at differential equations with $g(t) = 0$.

we call the differential equation **homogeneous** and when $g(x) \neq 0$ we call the differential equation **nonhomogeneous**.

So, let's start thinking about how to go about solving a constant coefficient, homogeneous, linear, higher order differential equation. Here is the general constant coefficient, homogeneous, linear, higher order differential equation.

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_o y = 0$$

All of the solutions in this example were in the form

$$y(x) = e^{rx}$$

To see if we are correct all we need to do is plug this into the differential equation and see what happens. So, let's get some derivatives and then plug in.

$$y'(x) = re^{rx}, \qquad y^{(2)}(t) = r^2 e^{rx}, \ldots, y^{(n)}(t) = r^n e^{rx}$$

$$a_n(r^n e^{rx}) + a_{n-1}(r^{n-1}e^{rx}) + \cdots + a_1 r e^{rx} + a_o e^{rx} = 0$$

$$e^{rx}(\, a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_o) = 0$$

This can be reduced further by noting that exponentials are never zero. Therefore, the value of r should be the roots of

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_o = 0.$$

This equation is typically called the **characteristic equation** for homogeneous equation.

Once we have these n roots, we have n solutions to the differential equation.

$$y_1(x) = e^{r_1 x}, \;\; y_2(x) = e^{r_2 x}, \ldots\ldots, y_n(x) = e^{r_n x}$$

You'll notice that we neglected to mention whether or not the n solutions listed above are in fact "nice enough" to form the general solution. This was intentional. We have three cases that we need to look at and this will be addressed differently in each of these cases.

So, what are the cases? The roots will have three possible forms. These are

1. Real, distinct roots, $r_1 \neq r_2$.
2. Complex root, $r_{1,2} = \lambda \pm \mu i$
3. Double roots, $r_1 = r_2 = r$.

For repeated roots we just add in a *t* for each of the solutions past the first one until we have a total of *k* solutions. Again, we will leave it to you to compute the Wronskian to verify that these are in fact a set of linearly independent solutions. Finally we need to deal with complex roots. The biggest issue here is that we can now have repeated complex roots for 4th order or higher differential equations. We'll start off by assuming that $r = \lambda \pm \mu i$ occurs only once in the list of roots. In this case we'll get the standard two solutions,

$$e^{\lambda x}\cos(\mu x) \qquad e^{\lambda x}\sin(\mu x)$$

Example.

Solve the following equation.

$$y^{(3)} - 5y'' - 22y' + 56y = 0$$

Answer :

The characteristic equation is

$$r^3 - 5r^2 - 22r + 56 = (r+4)(r-2)(r-7) = 0$$

Therefore we get

$$r_1 = -4, \qquad r_2 = 2, \quad r_3 = 7$$

So we have three real distinct roots here and so the general solution is

$$y(x) = c_1 e^{-4x} + c_2 e^{2x} + c_3 e^{7x}$$

Example.

Solve the following differential equation

$$2y^{(4)} + 11y^{(3)} + 18y'' + 4y' - 8y = 0$$

Answer :

The characteristic equation is

$$2r^4 + 11r^3 + 18r^2 + 4r - 8 = (2r - 1)(r + 2)^3 = 0$$

So, we have two roots here, $r_1 = \frac{1}{2}$ and $r_2 = -2$ which is multiplicity of 3. Remember that we'll get three solutions for the second root and after the first we add x's only the solution until we reach three solutions.

The general solution is then,

$$y(x) = c_1 e^{\frac{1}{2}x} + c_2 e^{-2x} + c_3 x e^{-2x} + c_4 x^2 e^{-2x}$$

Example.

Solve the following differential equation.

$$y^{(5)} + 12y^{(4)} + 104y^{(3)} + 408y'' + 1156y' = 0$$

Answer:

The characteristic equation is

$$r^5 + 12r^4 + 104r^3 + 408r^2 + 1156r = 0$$

So, we have one real root r = 0 and a pair of complex roots $r = -3 \pm 5 \pm i$ each with multiplicity 2. So, the solution for the real root is easy and for the complex roots we'll get a total of 4 solutions.

The general solution is

$$y(x) = c_1 + c_2 e^{-3x} \cos 5x + c_3 e^{-3x} \sin 5x + c_4 x e^{-3x} \cos 5x + c_5 x e^{-3x} \sin 5x$$

Exercise 24

Solve the problems below.

1. $y" - 5y' + 6y = 0$
2. $y^{(3)} - 3y" - y + 3y = 0$
3. $y" + 9y = 0$
4. $y^{(3)} - y" + y - y = 0$
5. $y^{(4)} + 8y" + 16y = 0$
6. $y^{(5)} - 2y^{(4)} + y^{(3)} = 0$
7. $y^{(4)} - 3y^{(3)\prime} - 2y" + 2y' + 12y = 0$

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27th, 28th Meeting : IV. Explicit Methods of Solving Higher Order Linear Differential Equations
- The Method of Undetermined Coefficients

We now consider the n-order linear differential equation which is nonhomogeneous

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_o y = G(x)$$

where $G(x) \neq 0$.

As we have studied before that the solution of nonhomogeneous equation above is in the form

$$y = y_c + y_p$$

Where y_c is the general solution for the corresponding homogeneous equation and y_p is the particular solution.

How to determine the particular solution, can be done with Undetermined Coefficient Method and Variation Parameters Method.

IV.A Undetermined Coefficient Method

If $G(x)$ is an exponential function, polynomial, sine, cosine, sum/difference of one of these and/or a product of one of these then we guess the form of a particular solution. We then plug the guess into the differential equation, simplify and set the coefficients equal to solve for the constants. The one thing that we need to recall is that we first need the complimentary solution prior to making our guess for a particular solution. If any term in our guess is in the complimentary solution then we need to multiply the portion of our guess that contains that term by a *x*.

Example.

Solve the following differential equation

$$y^{(3)} - 12y" + 48y' - 64y = 12 - 32e^{-8x} + 2e^{4x}$$

Answer :

The characteristic equation corresponds to the homogeneous part is

$$r^3 - 12r^2 + 48r - 64 = (r-4)^3 = 0$$

So, the root is 4 of multiplicity 3.

Therefore the complimentary solution is

$$y_c(x) = c_1e^{4x} + c_2xe^{4x} + c_3x^2e^{4x}$$

Based on the right side of the equation $12 - 32e^{-8x}8x + 2e^{4x}$, our guess for a particular solution is

$$Y_p(x) = A + Be^{-8x}8x + Ce^{4x}$$

Notice that the last term in our guess is the complimentary solution, so we'll need to add one at least t to the third term in our guest. Also notice that multiplying in the third term by either x or x^2 will result in a new term that is still in the complimentary solution. So, we need to multiply the third term by x^3 in order to get a term that is not contained in the complimentary solution.

The final guess is then

$$Y_p(x) = A + Be^{-8x}8x + Cx^3e^{4x}$$

Substitute this function to the equation by taking it's first, second and third derivative then set the coefficients equal we get,

$$A = -\frac{3}{16}, \qquad B = \frac{1}{54}, \quad C = \frac{1}{3}$$

The general solution is then

$$y(x) = c_1e^{4x} + c_2xe^{4x} + c_3x^2e^{4x} - \frac{3}{16} + \frac{1}{54}e^{-8x}8x + \frac{1}{3}x^3e^{4x}$$

TABLE 4.1

	UC function	UC set
1	x^n	$\{x^n, x^{n-1}, x^{n-2}, \ldots, x, 1\}$
2	e^{ax}	$\{e^{ax}\}$
3	$\sin(bx + c)$ or $\cos(bx + c)$	$\{\sin(bx + c), \cos(bx + c)\}$
4	$x^n e^{ax}$	$\{x^n e^{ax}, x^{n-1}e^{ax}, x^{n-2}e^{ax}, \ldots, xe^{ax}, e^{ax}\}$
5	$x^n \sin(bx + c)$ or $x^n \cos(bx + c)$	$\{x^n \sin(bx + c), x^n \cos(bx + c), x^{n-1} \sin(bx + c), x^{n-1} \cos(bx + c), \ldots, x \sin(bx + c), x \cos(bx + c), \sin(bx + c), \cos(bx + c)\}$
6	$e^{ax} \sin(bx + c)$ or $e^{ax} \cos(bx + c)$	$\{e^{ax} \sin(bx + c), e^{ax} \cos(bx + c)\}$
7	$x^n e^{ax} \sin(bx + c)$ or $x^n e^{ax} \cos(bx + c)$	$\{x^n e^{ax} \sin(bx + c), x^n e^{ax} \cos(bx + c), x^{n-1} e^{ax} \sin(bx + c), x^{n-1} e^{ax} \cos(bx + c), \ldots, xe^{ax} \sin(bx + c), xe^{ax} \cos(bx + c), e^{ax} \sin(bx + c), e^{ax} \cos(bx + c)\}$

Exercise 27

Find the general solution of the differential equations below

1. $y'' - 2y' - 8y = 4e^{2x} - 21e^{-3x}$
2. $y'' - 3y + 2y = 4x^2$
3. $y'' + 2y' + 5y = 6 \sin 2x + 7 \cos 2x$
4. $y'' + 2y' + 2y = 10 \sin 4x$
5. $y'' + y' - 2y = 6e^{-2x} + 3e^x - 4x^2$
6. $y^{(3)} - 4y'' + 5y' - 2y = 3x^2\, e^x - 7e^x$

29th, 30th Meeting : IV. Explicit Methods of Solving Higher Order Linear Differential Equations
- Variation of Parameters

IV.B Variation of Parameters Methods

The method of variation of parameters involves trying to find a set of new functions, $u_1(x), u_2(x), \ldots, u_n(x)$ so that,

$$Y(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \ldots + u_n(x)$$

will be a solution to the nonhomogeneous differential equation. In order to determine if this is possible, and to find the $u_i(x)$ if it's possible, we'll need a total of n equations involving the unknown functions that we can solve.

Suppose

$$W(y_1, y_2, \ldots, y_n)(x) = \begin{vmatrix} y_1(x) & y_2(x) & y_3(x) & \cdots & y_n(x) \\ y'_1(x) & y'_2(x) & y'_3(x) & & y'_n(x) \\ \vdots & & \vdots & & \vdots \\ y^{(n-1)}{}_1(x) & y^{(n-1)}{}_2(x) & y^{(n-1)}{}_3(x) & \cdots & y^{(n-1)}{}_n(x) \end{vmatrix}$$

Let W_i represent the determinant we get by replacing the i^{th} column of the Wronskian with the column $(0,0,0,\ldots,0,1)$. For example

$$W_1 = \begin{vmatrix} 0 & y_2(x) & y_3(x) & \cdots & y_n(x) \\ 0 & y'_2(x) & y'_3(x) & & y'_n(x) \\ \vdots & & \vdots & & \vdots \\ 1 & y^{(n-1)}{}_2(x) & y^{(n-1)}{}_3(x) & \cdots & y^{(n-1)}{}_n(x) \end{vmatrix}$$

the solution to the system can then be written as,

$$u_1(x) = \int \frac{g(x)W_1(x)}{W(x)}\,dx, \quad u_2(x) = \int \frac{g(x)W_2(x)}{W(x)}\,dx, \ldots, u_n(x) = \int \frac{g(x)W_n(x)}{W(x)}\,dx$$

Finally, the particular solution is given by

$$Y_p(x) = y_1(x)\int \frac{g(x)W_1(x)}{W(x)}\,dx + y_2(x)\int \frac{g(x)W_2(x)}{W(x)}\,dx + \ldots + y_n(x)\int \frac{g(x)W_n(x)}{W(x)}\,dx$$

Example.

Solve the following differential equation

$$y^{(3)} - 2y'' - 21y' - 18y = 3 + 4e^{-x}$$

Answer :

The characteristic equation corresponds to the homogeneous equation is

$$r^3 - 2r^2 - 21r - 18 = 0 \rightarrow r = -3, r = -1, r = 6$$

Here we get

$$y_c(x) = c_1 e^{-3x} + c_2 e^{-x} + c_3 e^{6x}$$

Several determinant that we should count are

$$W = \begin{vmatrix} e^{-3x} & e^{-x} & e^{6x} \\ -3e^{-3x} & -e^{-x} & 6e^{6x} \\ 9e^{-3x} & e^{-x} & 36e^{6x} \end{vmatrix} = 126e^{2x}$$

$$W_1 = \begin{vmatrix} 0 & e^{-x} & e^{6x} \\ 0 & -e^{-x} & 6e^{6x} \\ 1 & e^{-x} & 36e^{6x} \end{vmatrix} = 7e^{5x}$$

$$W_2 = \begin{vmatrix} e^{-3x} & 0 & e^{6x} \\ -3e^{-3x} & 0 & 6e^{6x} \\ 9e^{-3x} & 1 & 36e^{6x} \end{vmatrix} = -9e^{3x}$$

$$W_3 = \begin{vmatrix} e^{-3x} & e^{-x} & 0 \\ -3e^{-3x} & -e^{-x} & 0 \\ 9e^{-3x} & e^{-x} & 1 \end{vmatrix} = 2e^{-4x}$$

Here we get

$$u_1(x) = \int \frac{(3 + 4e^{-x})(7e^{5x})}{(126e^{2x})} dx = \frac{1}{18}(e^{3x} + 2e^{2x})$$

$$u_2(x) = \int \frac{(3 + 4e^{-x})(-9e^{3x})}{(126e^{2x})} dx = -\frac{1}{14}(3e^{x} + 4x)$$

$$u_3(x) = \int \frac{(3 + 4e^{-x})(2e^{-4x})}{(126e^{2x})} dx = \frac{1}{63}\left(-\frac{1}{2}e^{-6x} - \frac{4}{7}e^{-7x}\right)$$

The general solution is then,

$$y(x) = c_1 e^{-3x} + c_2 e^{-x} + c_3 e^{6x} - \frac{1}{6} + \frac{5}{49}e^{-x} - \frac{2}{7}e^{-x}$$

Exercise 29

Solve the differential equations below.

1. $y" - 2y' - 8y = 4e^{2x} - 21e^{-3x}$
2. $y" - 3y + 2y = 4x^2$
3. $y^{''} + 2y^{'} + 5y = 6\sin 2x + 7\cos 2x$
4. $y^{''} + 2y^{'} + 2y = 10\sin 4x$
5. $y^{''} + y^{'} - 2y = 6e^{-2x} + 3e^x - 4x^2$
6. $y^{(3)} - 4y^{''} + 5y^{'} - 2y = 3x^2\, e^x - 7e^x$

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31st Meeting : EXERCISES

Question 1 : Solve the differential equation below.

$$\frac{dy}{dx} = \frac{x+3y-5}{x-y-1}$$

Question 2 : Sketch for the graph of solution of without finding it's solution explicitly and give some reasons.

$$y' = (2-y)y\,, \qquad y(0) = \tfrac{1}{2}$$

Question 3 : Find the general solution of differential equation $y'' + 6y' + 9 = 0$.

Question 4 : Find the general solution of differential equation below.

$$ty' + 2y = \sin t \qquad ,t > 0$$

Question 5 : Find the general solution of the differential equation below.

$$(x^2 + y)dx + (x + e^y)dy = 0$$

Question 6 : Find the general solution of the second order linear differential equation below.

$$y'' + 2y' = 3 + 4\sin 2t$$