STATE UNIVERSITY OF-YOGYAKARTA
FACULTY OF-MATHEMATICS AND NATURAL SCIENCE
DEPARTMENT OF MATHEMATICS EDUCATION

Fourth Meeting : I. First Order Differential Equations for Which Exact Solutions are obtainable
- Standard forms of First Order Differential Equations
- Exact Equation

In this chapter, we will discuss the solution of first order differential equation. The main idea is to check whether it is an exact or nonexact. For Nonexact first order differential equation, we should use integrating factor to transform nonexact differential equation into the new differential equation which is exact.

II.A Standard forms of First Order Differential Equations

First order differential equation can be expressed as derivative form

$$\frac{dy}{dx} = f(x, y) \ldots \ldots (2.1)$$

or as differential form

$$M(x, y)\, dx + N(x, y)\, dy = 0 \ldots \ldots (2.2)$$

Definition 2.1

Let F be a function of two real variables such that F has continuous first partial derivatives in a domain D. The total differential dF of the function F is defined by the formula

$$dF(x,y) = \frac{\partial F(x, y)}{\partial x}\, dx + \frac{\partial F(x,y)}{\partial y}\, dy$$

For all $(x, y) \in D$.

Definition 2.2

In a domain D, if there exists a function F of two real variables such that expression

$$M(x,y)dx + N(x,y)dy$$

equals to the total differential $dF(x,y)$ for all $(x,y) \in D$ then this expression is called an exact differential.

In other words, the expression above is an exact differential in D if there exists a function F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \text{ and } \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

for all $(x,y) \in D$.

EXACT EQUATION

If $M(x,y)dx + N(x,y)dy$ is an exact differential, then the differential equation

$$M(x,y)\ dx + N(x,y)\ dy = 0$$

is called **an exact differential equation.**

The following theorem is used to identify whether a differential equation is an exact or not.

Theorem

Consider the differential equation

$$M(x,y)\ dx + N(x,y)\ dy = 0$$

Where M and N have continuous first partial derivatives at all points (x,y) in a rectangular domain D.

1. If the differential equation is exact in D, then

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

For all $(x,y) \in D$

2. Conversely, if

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

For all $(x,y) \in D$, then the differential equation is exact in D.

Example 2.1

Determine the following differential equation is exact or nonexact.

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0$$

Answer :

The differential equation is expressed in derivative form. Transformed into differential form, we get

$$(2xy - 9x^2)\ dx\ (2y + x^2 + 1)\ dy = 0$$

Suppose $M(x, y) = 2xy - 9x^2$ and $N(x,y) = 2y + x^2 + 1$, then the differential equation above is exact if and only if

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}.$$

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial(2xy - 9x^2)}{\partial y} = 2x$$

While

$$\frac{\partial N(x,y)}{\partial x} = \frac{\partial x\ (2y + x^2 + 1)}{\partial x} = 2x$$

Because $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$, then the differential equation above is an exact.

Exercises 5.

Determine whether it is an exact differential equation or not. Explained!

1. $[1 + \ln(xy)]\ dx + \frac{x}{y}\ dy = 0$
2. $x^2y\ dx - (xy^2 + y^3)\ dy = 0$
3. $(y + 3x^2)\ dx + x\ dy = 0$
4. $[\cos(xy) - xy\sin(xy)]\ dx - x^2\sin(xy)\ dy = 0$
5. $ye^{xy}\ dx + (2y - xe^{xy})\ dy = 0$