MEETING: 17-18

Learning Objectives: Students are able to

1. explain analytic functions
2. explain harmonic functions
3. determine a harmonic conjugate

1. Analytic Functions

Definition

(a). A function f of the complex variable z is *analytic* at point z_0 if its derivative exists not only at z_0 but at each point z in some neighborhood of z_0.

(b). An *entire* function is a function that is analytic at each point in the entire plane.

(c). If a functions fails to be analytic at a point z_0 but is analytic at some point in every neighborhood of z_0 is called a *singular point* or *singularity of the function*.

Example: the function

$$f(z) = \frac{1}{z} \text{ then } f'(z) = -\frac{1}{z^2}$$

We can see that f is analytic except in the point $z = 0$ in the finite plane. Moreover, $z = 0$ is called a singular point.

2. Harmonic Functions

A real-valued functions h of two real variables x and y is said to be *harmonic* in a given domain of the xy plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$h_{xx}(x, y) + h_{yy}(x, y) = 0 \tag{21}$$

known as Laplace's equation.

Example

The function $H(x,y) = e^{-y}\sin x$ is harmonic since $H_x = e^{-y}\cos x$, $H_{xx} = -e^{-y}\sin x$, $H_y = -e^{-y}\sin x$, $H_{yy} = e^{-y}\sin x$ and $H_{xx}(x,y) + H_{yy}(x,y) = 0$

3. Harmonic Conjugate

If two given functions *u* and *v* are harmonic in a domain *D* and their first partial derivatives satisfy the Cauchy-Riemann equations throughout *D*, we say that *v* is a *harmonic conjugate* of *u.* We now illustrate one method of obtaining a harmonic conjugate of a given harmonic function.

The function

$$u(x,y) = y^3 - 3x^2 y.$$

To find a harmonic conjugate $v(x,y)$, we note that $u_x(x,y) = -6xy$.

So, in view of the condition $u_x = v_y$ we may write $v_y = -6xy$.

Holding *x* fixed and integrating both sides of this equation with respect to *y,* we find that

$$v(x,y) = -3xy^2 + g(x)$$

where $g(x)$ is arbitrary function of *x*. Since the condition $u_y = -v_x$ must hold, it follows that

$$3y^2 - 3x^2 = 3y^2 - g'(x)$$

So, $g'(x) = 3x^2$, and this means that $g(x) = x^3 + c$, where *c* is an arbitrary real number. Hence the function

$$v(x,y) = x^3 - 3xy^2 + c$$

is a harmonic conjugate of our function. The corresponding analytic function is

$$f(z) = \left(y^3 - 3x^2 y\right) + i\left(x^3 - 3xy^2 + c\right)$$

It is easily verified that

$$f(z) = i\left(z^3 + c\right)$$

Exercises:

1. In each case, determine the singular points of the functions and state why the function is analytic everywhere except at those points

a. $f(z)=\dfrac{z+1}{z(z^2+2)}$ b. $f(z)=\dfrac{z}{z^2-2iz+3}$

2. Show that the following functions are entire
   a. $f(z)=\cosh x\sin y-i\sinh x\cos y$
   b. $g(z)=\cosh x\cos y+i\sinh x\sin y$
3. Let *a, b* and *c* be real constants. Determine a relation among the coefficients that will guarantee that the function $f(x,y)=ax^2+bxy+cy^2$ is harmonic.
4. Show that $u(x,y)$ is harmonic in some domain and find a harmonic conjugate $v(x,y)$ when
   a. $u(x,y)=2x(1-y)$
   b. $u(x,y)=2x-x^3+3xy^2$
5. Let $u_1(x,y)=x^2-y^2$ and $u_2(x,y)=x^3-3xy^2$. Show that u_1 and u_2 are harmonic functions and that their product $u_1(x,y).u_2(x,y)$ is not a harmonic function.