MEETING: 19-20

Learning Objectives: Students are able to

1. explain exponential functions
2. explain trigonometric functions

1. The Exponential Function

Definition

The exponential function of complex analysis is thus defined for all z by means of the equation

$$e^z = e^x\left(\cos y + i\sin y\right), \tag{22}$$

where $z = x + iy$.

When z is the pure imaginary number $i\theta$, we have $e^{i\theta} = \cos\theta + i\sin\theta$. This is called Euler's formula. For every $z = x + iy$, it is enables us to express e^z in the more compact form $e^z = e^x e^{iy}$.

Properties:

For every $z, z_1, z_2 \in \mathbb{C}$,

a. $e^{z_1}e^{z_2} = e^{z_1+z_2}$

b. $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$

c. $\dfrac{d}{dz}e^z = e^z$

d. $\left|e^z\right| = e^x$

2. The Trigonometric Function

Definition:

$\forall z \in \mathbb{C}$, $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$.

Properties:

a. $\cos(-z) = \cos z$

b. $\sin(-z) = -\sin z$

c. $\dfrac{d}{dz}\sin z = \cos z$

d. $\dfrac{d}{dz}\cos z = -\sin z$

e. $\sin(z_1+z_2)=\sin z_1\cos z_2+\cos z_1\sin z_2$

f. $\cos(z_1+z_2)=\cos z_1\cos z_2-\sin z_1\sin z_2$

g. $\sin^2 z+\cos^2 z=1$

h. $\sin 2z=2\sin z\cos z$

i. $\cos 2z=\cos^2 z-\sin^2 z$

j. $\sin z=\sin x\cosh y+i\cos x\sinh y$, $\forall z=x+iy$

k. $\cos z=\cos x\cosh y-i\sin x\sinh y$, $\forall z=x+iy$

l. $\sin z=0 \Leftrightarrow z=n\pi$, $n=0,\pm1,\pm2,\dots$

m. $\cos z=0 \Leftrightarrow z=\dfrac{\pi}{2}+n\pi$, $n=0,\pm1,\pm2,\dots$

n. $\tan z=\dfrac{\sin z}{\cos z}$

Exercises

1. Show that

   a. $e^{(2\pm3\pi i)}=-e^2$ b. $e^{\left(\frac{2+\pi i}{4}\right)}=\sqrt{\frac{e}{2}}(1+i)$

2. Show that $e^{z_1}e^{z_2}=e^{z_1+z_2}$

3. Find all solution of

   a. $e^z=1$ b. $e^z=i$ c. $e^z=1+i$

4. Let n be a positive integer. Show that

   a. $(e^z)^n=e^{nz}$ b. $\dfrac{1}{(e^z)^n}=e^{-nz}$

5. Show that $\overline{\sin z}=\sin\overline{z}$

6. If $\cos z=2$, determine the value of $\cos 2z$.

7. Find all complex numbers z such that $\sin z=2$.

8. Express the following quantities in $u+iv$ form

   a. $\cos(1+i)$

b. $\sin\left(\dfrac{\pi+4i}{4}\right)$

9. Find the derivatives of the following

a. $\sin(1/z)$

$z \tan z$