MEETING: 21-22

Learning Objectives: Students are able to

1. explain hyperbolic functions
2. explain logarithmic functions

1. The Hyperbolic Function

Definition:

$$\forall z \in \mathbb{C}, \sinh z = \frac{e^z - e^{-z}}{2}, \cosh z = \frac{e^z + e^{-z}}{2}.$$

Furthermore,

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \frac{d}{dz}\sinh z = \cosh z,$$

$$\coth z = \frac{\cosh z}{\sinh z}, \qquad \frac{d}{dz}\cosh z = \sinh z,$$

$$\sec hz = \frac{1}{\cosh z}, \qquad \frac{d}{dz}(\tanh z) = \sec h^2 z,$$

$$\csc hz = \frac{1}{\sinh z}, \qquad \frac{d}{dz}(\sec hz) = -\sec hz \tan z.$$

The relationship between hyperbolic function and trigonometric function:

a. $\cosh(iz) = \cos z$

b. $\cos(iz) = \cosh z$

c. $-i\sinh(iz) = \sin z$

d. $-i\sin(iz) = \sinh z$

Some of the most frequently used identities involving hyperbolic sine and cosine functions are

$$\sinh(-z) = -\sinh z,$$

$$\cosh(-z) = \cosh z,$$

$$\cosh^2 z - \sinh^2 z = 1,$$

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2,$$

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2.$$

$$\sinh z = \sinh x \cos y + i\cosh x \sin y, \ \forall z = x + iy,$$

$$\cosh z = \cosh x \cos y + i\sinh x \sin y, \ \forall z = x + iy,$$

$$|\sinh z|^2 = \sinh^2 x + \sin^2 y, \ \forall z = x + iy,$$

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y, \ \forall z = x + iy,$$

2. The Logarithmic Function

For every $z = re^{i\theta}$,

$$\log z = \ln r + i(\theta + 2n\pi), n \in \mathbb{Z}.$$

The principal value of $\log z$ is denoted by $Log\, z$, define $Log\, z = \ln r + i\theta, -\pi < \theta \le \pi$.

Example

$\log(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right), n = 0, \pm 1, \pm 2, \ldots$ and the principal value is $Log(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4}\right)$.

Properties:

Let $z, z_1, z_2 \in \mathbb{C}$

a. $\frac{d}{dz}\log z = \frac{1}{z}$

b. $e^{\log z} = z$

c. $\log(z_1 z_2) = \log z_1 + \log z_2$

d. $\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$

Exercises:

1. Show that $\sin(iz) = i\sinh z$
2. Find all values of z for which the following equations hold

   a. $\sinh z = \frac{i}{2}$ b. $\cosh z = 1$
3. Show that $\sinh(z + i\pi) = -\sinh z$
4. Find the principal value for the following

   a. $Log(ie^2)$ b. $Log(i\sqrt{2} - \sqrt{2})$
5. Find all the values of $\log z$ for the $\log(-3)$
6. Find all the values of z for which the following equations hold:

   a. $Log\, z = 1 - \frac{i\pi}{4}$ b. $e^z = -ie$