BUILDING THE REAL NUMBER SYSTEM

The Real Number System consists of:

- Natural Numbers
- Whole Numbers
- Integers
- Fractions
- Rational Numbers
- Irrational Numbers
- Real Numbers

Postulate for Whole Numbers:

1.	$a + b = b + a$	Commutative Law of Addition
2.	$(a + b) + c = a + (b + c)$	Associative Law of Addition
3.	$a(b + c) = a \cdot b + a \cdot c$	Distributive Law
4.	$ab = ba$	Commutative Law of Multiplication
5.	$(ab)c = a(bc)$	Associative Law of Multiplication
6.	$a + 0 = 0 + a = a$	Zero Property

Commutative:

$$\underbrace{\text{II} + \text{III}}_{\text{combining 2 and 3}} = \text{IIIII}$$

$$\underbrace{\text{III} + \text{II}}_{\text{combining 3 and 2}} = \text{IIIII}$$

Associative:

$$\underbrace{\underbrace{(\text{II} + \text{III})}_{2+3} + \text{IIII}}_{(2+3)+4} = (\text{IIIII}) + \text{IIII} = \underbrace{\text{IIIIIIIII}}_{\text{result}}$$

Next, examine $2 + (3 + 4)$

$$\underbrace{\text{II} + \underbrace{(\text{III} + \text{IIII})}_{3+4}}_{2+(3+4)} = \text{II} + (\text{IIIIIII}) = \underbrace{\text{IIIIIIIII}}_{\text{result}}$$

Distributive:

$$\underbrace{(\text{III} + \text{IIII}) + (\text{III IIII})}_{2(3+4)} \overset{\text{upon rearranging}}{=} \underbrace{(\text{III} + \text{III})}_{2\cdot 3 +} + \underbrace{(\ \text{IIII} + \text{IIII})}_{2\cdot 4}$$

You may be wondering if one can rigorously *prove* that these laws are true. That way, we wouldn't have to just accept them. The surprising answer is, "Yes," and a mathematician, Guisseppe Peano (1852–1932) did prove them by just assuming more basic facts about numbers, together with the principle of mathematical induction. He and others developed the entire real number system from scratch and proved all the laws that we accept. Thus, they put this area of mathematics on a solid foundation. It is a beautiful piece of work, but beyond the scope of this book, since it requires a very detailed analysis that would take at least a semester to do completely. So, we will stick to what people observed, and continue to develop the number system intuitively, just as human beings did. Ultimately, we can rest assured that mathematicians have proven these rules.

Negative Numbers and Their Properties

7. For every whole number a, there is, by creation, a "number," denoted by $-a$ such that $a + (-a) = 0$.

Additive Inverse Property

Prove that:
There can only be one additive inverse of a number

- Assume that there are two numbers that are additive inverses of a given number
- Argue that they must be equal
- Suppose the additive inverses of *x* is *a* and *b*,

then $x + a = 0$ and $x + b = 0$

Now,

$$
\begin{aligned}
a &= a + 0 && \text{(Rule 6 above)} \\
&= a + (x + b) && \text{(By equation (6.12))} \\
&= (a + x) + b && \text{(Associative law)} \\
&= (x + a) + b && \text{(Commutative law)} \\
&= 0 + b && \text{(By equation (6.11))} \\
&= b && \text{(Zero property)}
\end{aligned}
$$

One of the typical questions that teacher candidates are asked when going on a job interview for a mathematics teaching position is how to explain to students that a negative number times a negative number is a positive number. What would you say, if you were asked this question on an interview?

Prove that using real life context.

Prove using mathematics by following steps:

(a) *The equation* $x + x = x$ *has only one solution, namely* $x = 0$.

(b) *If* a *represents any number, then* $a(0) = 0$.

(c) $(-a)(b) = -(ab)$. *In particular, a negative times a positive is a negative.*

(d) $(-a)(-b) = ab$. *In particular, a negative times a negative is a positive.*

Proof. (a) Start with $x + x = x$ and rewrite this as $(x + x) = x$. Add $-x$ to both sides to get

$$(x + x) + (-x) = x + (-x).$$

Use the Associative Law to rewrite this as

$$x + (x + (-x)) = x + (-x).$$

Use rule 7 above to rewrite this as

$$x + 0 = 0.$$

Finally, use the rule 6 above to rewrite this as

$$x = 0.$$

Thus, if $x + x = 0$, we have that $x = 0$.

(b) Since $0 + 0 = 0$ by rule 6 above, with $a = 0$, we have,

$$a(0) = a(0 + 0).$$

Distributing, we get

$$a(0) = a(0) + a(0).$$

Now, calling $a(0) = x$, this becomes

$$x + x = x.$$

And now by part (a),

$$x = 0. \tag{6.13}$$

But $x = a(0)$. Thus, equation (6.13) says, $a(0) = 0$.

(c) We now know that $a(0) = 0$. Rewrite this as $a((-b) + b) = 0$. Distribute to get

$$a(-b) + ab = 0. \tag{6.14}$$

Equation (6.14) says that $a(-b)$ is an additive inverse of ab, since they sum to zero. But there is only one additive inverse of ab and that is $-(ab)$. Thus,

$$a(-b) = -(ab). \tag{6.15}$$

(d) We already know that any number times 0 is 0. Thus, $(-a)(0) = 0$. Rewrite this as $(-a)(-b + b) = 0$. Distributing we get,

$$(-a)(-b) + (-a)b = 0. \tag{6.16}$$

By equation (6.15), equation (6.16) reduces to

$$(-a)(-b) + (-(ab)) = 0. \tag{6.17}$$

Now, equation (6.17) says that $(-a)(-b)$ is an additive inverse of $-(ab)$. But so is ab. Since the additive inverse of $(-a)(-b)$ is unique, $(-a)(-b) = ab$.

Fractions and Their Properties

Addition of Fractions

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}.$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad+bc}{bd}.$$

Multiplication of Fractions

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Division of Fractions

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}.$$

Golden Rule of Fractions

Golden Rule of Fractions: The numerator and denominator of a fraction can both be multiplied by the same nonzero quantity, and we will get an equivalent fraction. In symbols, the Golden Rule says that, if a, b, and k are positive numbers and $b \neq 0$, then

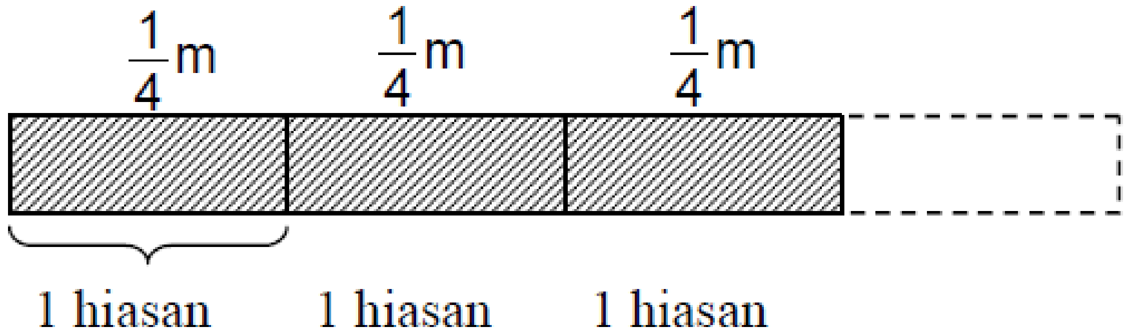
$$\frac{a}{b} = \frac{ak}{bk} \quad \text{if} \quad k \neq 0. \tag{6.8}$$

If the Golden Rule is to be true for all fractions, then we can multiply the numerator and denominator of this complex fraction by the same quantity, $\frac{5}{2}$. This yields

$$\frac{\frac{1}{3}}{\frac{2}{5}} = \frac{\frac{1}{3} \cdot \frac{5}{2}}{\frac{2}{5} \cdot \frac{5}{2}} = \frac{\frac{1}{3} \cdot \frac{5}{2}}{1} = \frac{1}{3} \cdot \frac{5}{2}.$$

Kakak mempunyai $\frac{3}{4}$ m pita yang akan dibuat hiasan, dan setiap hiasan memerlukan $\frac{1}{4}$ m pita. Hiasan yang dapat dibuat ada sebanyak ….

Dalam kalimat matematika adalah $\frac{3}{4} : \frac{1}{4} = \ldots$

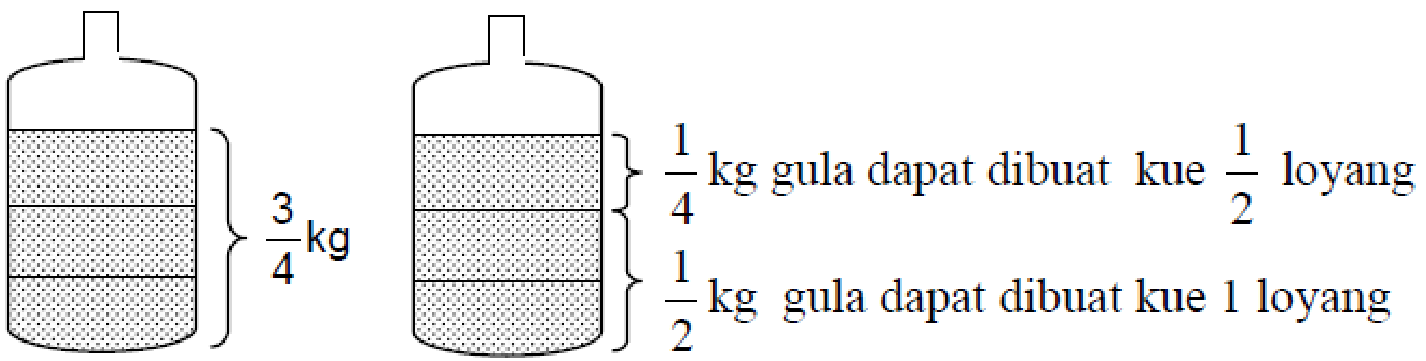


Dari gambar tampak bahwa ada 3 hiasan yang dapat dibuat dari $\frac{3}{4}$ m pita. Jadi $\frac{3}{4} : \frac{1}{4} = 3$.

Ibu mempunyai gula $\frac{3}{4}$ kg yang akan dibuat kue. Setiap loyang kue memerlukan $\frac{1}{2}$ kg gula. Banyaknya kue yang dapat dibuat …. loyang

Kalimat matematika dari soal di atas adalah $\frac{3}{4} : \frac{1}{2} = \ldots$

Gula yang ada digambarkan ditempatkan pada kantong sebagai berikut.



Jadi dari gambar terlihat bahwa $\frac{3}{4}$ kg gula dapat dibuat kue $1\frac{1}{2}$ loyang, dan kalimat matematika yang bersesuaian adalah $\frac{3}{4} : \frac{1}{2} = 1\frac{1}{2} = \frac{3}{2}$.

Jadi Ibu dapat membuat kue sebanyak $1\frac{1}{2}$ loyang.

Dari kedua contoh di atas diperoleh:

$$\left.\begin{cases} \text{hasil pembagian } \dfrac{3}{4} : \dfrac{1}{2} = \dfrac{3}{2} \\ \text{hasil perkalian } \dfrac{3}{4} \times \dfrac{2}{1} = \dfrac{6}{4} = \dfrac{3}{2} \end{cases}\right\} \text{ sehingga } \frac{3}{4} : \frac{1}{2} = \frac{3}{4} \times \frac{2}{1}$$

$$\left.\begin{cases} \text{hasil pembagian } \dfrac{5}{6} : \dfrac{1}{3} = \dfrac{5}{2} \\ \text{hasil perkalian } \dfrac{5}{6} \times \dfrac{3}{1} = \dfrac{15}{6} = \dfrac{5}{2} \end{cases}\right\} \text{ sehingga } \frac{5}{6} : \frac{1}{3} = \frac{5}{6} \times \frac{3}{1}$$

Dari uraian di atas dapat disimpulkan secara umum bahwa:

$$\boxed{\frac{a}{b} : \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}}$$

Prove that the commutative, associative, and distributive laws hold for fractions.

Why we don't allow to divide by 0?

We have a = b = 1

$$a = b$$

$$ab = b^2$$

$$ab - a^2 = b^2 - a^2$$

$$a\,(b - a) = (b - a)\,(b + a)$$

$$a = b + a$$

$$1 = 2$$

What's wrong with this proof?

Rational Numbers and Irrational Numbers

Real numbers can be modeled perfectly using the number line

The next two results show that rationals and irrationals are everywhere. However, since in this book, we are not developing the real number system completely, we have to make use of some facts about real numbers that are intuitive. Here are the facts we accept: (a) The fraction $\frac{1}{n}$ can be made as small as we want by taking n large. (Thus, if n is one million, this fraction is $\frac{1}{1{,}000{,}000}$ which is small.) (b) Between any two numbers that differ by 1, there lies some integer. That is, for any number a, there is always some integer k, that satisfies $a < k \leq a + 1$. For example, if $a = 3.5$, this last statement says that between 3.5 and 4.5 there is an integer k, specifically the integer 4. If $a = 2$, the above statement says that there is an integer k that satisfies $2 < k \leq 3$. Obviously, that integer k is 3.

Prove that:

(1) Between every two real numbers there is a rational number.

(2) Between every two rational numbers there is an irrational number.

Proof. (1) Suppose x and y are any two real numbers, and that $x < y$. This implies that $y - x > 0$. Since $\frac{1}{n}$ can be made as small as we want, there is some positive number n that makes $\frac{1}{n} < y - x$. Let us take the smallest such n. Since the numbers nx and $nx + 1$ differ by 1, we know there is some number k that satisfies $nx < k \leq nx + 1$. Divide this inequality by n to get $x < \frac{k}{n} \leq x + \frac{1}{n}$. But, since we know that $\frac{1}{n} < y - x$ (notice the strict inequality), the previous inequality can be written as $x < \frac{k}{n} < x + (y - x)$, or just as $x < \frac{k}{n} < y$.

We have found that between *any* two real numbers x and y there is a rational number $\frac{k}{n}$, so we have proven the first part of the theorem.

Proof. (2) Take x and y rational and suppose that $x < y$. Multiply both sides of this inequality by $\sqrt{2}$ to get $\sqrt{2}x < \sqrt{2}y$. Now, by part (1) of the theorem, there is a rational number k between the two real numbers $\sqrt{2}x$ and $\sqrt{2}y$. That is, there is a rational number k such that $\sqrt{2}x < k < \sqrt{2}y$. Divide this inequality by $\sqrt{2}$. to get $x < \frac{k}{\sqrt{2}} < y$. And since (Student Learning Opportunity 7) a rational number divided by an irrational number is irrational, we have found an irrational number, $\frac{k}{\sqrt{2}}$, between x and y. ■