Section 1.3. Homogeneous Equations

A system equations in the variables $x_1, x_2, x_3, ..., x_n$ is called homogeneous if all the constant terms are zero, that is, if each equation of the system has form:

$$a_1x_1 + a_2x_2 + a_3x_3 + ... + a_nx_n = 0$$

Clearly $x_1 = 0, x_2 = 0, x_3 = 0, ..., x_n = 0$ is a solution to such a system, it is called the trivial solution. Any solution in which at least one variable has a nonzero value is called a nontrivial solution. Our chief goal in this section is to give a useful condition for a homogeneous system to have nontrivial solutions. The following example is instructive.

Example 10 Show that the following homogeneous system has nontrivial solution:

$$x_1 - x_2 + 2x_3 + x_4 = 0$$

$$2x_1 + 2x_2 \qquad - x_4 = 0$$

$$3x_1 + x_2 + 2x_3 + x_4 = 0$$

Solution

The reduction of the augmented matrix to reduced row-echelon form is outlined below:

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 2 & 2 & 0 & -1 & 0 \\ 3 & 1 & 2 & 1 & 0 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 4 & -4 & 1 & 0 \\ 0 & 4 & -4 & -2 & 0 \end{array}\right] \xrightarrow[\frac{1}{4}R_3]{\frac{1}{4}R_2}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 1 & -1 & \frac{1}{4} & 0 \\ 0 & 1 & -1 & -\frac{1}{2} & 0 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 1 & -1 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \end{array}\right] \xrightarrow{-2R_3}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 1 & -1 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_2 - \frac{1}{4}R_3]{R_1 - R_3}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The leading variables are x_1, x_2, x_4, so x_3 is assign as a parameter, say $x_3 = t$. Then the general solution is $x_1 = -t, x_2 = t, x_3 = t, x_4 = 0$. Hence, taking $t = 1$, we get a nontrivial solution.

The existence of a nontrivial solution in Example 1 is ensured by the presence of a parameter in the solution. This is due to the fact that there is a non leading variable (x_3 in this case). But there must be a non leading variable here because there are four variable and only three equations (and hence at most three leading variables). This discussion generalizes to a proof of the following useful theorem.

Theorem 1 If a homogeneous system of linear equations has more variable than equation then it has a nontrivial solution (in fact, infinitely many)

Proof:

Suppose there are *m* equations in *n* variables where $n > m$, and let R denote the reduced row echelon form of the augmented matrix. If there are *r* leading variables, there are *n-r* non leading variables, and so *n-r* parameters. Hence, it suffices to show that $r<n$. But $r \le m$ because R has *r* leading 1's and m rows and $m<n$ by hypothesis.

Note that the converse of Theorem 1 is not true: if a homogeneous system has nontrivial solutions, it need not have more variables than equations.

Exercises 1.3

Ohm's Law The current I and a voltage drop V across a resistance R are related by the equation $V = RI$

Kirchhoff's Law
1. (Junction Rule):
   The current flow into a junction equals the current flow out of that junction.
2. (Circuit Rule)
   The algebraic sum of the voltage drops (due to resistance) around any closed circuit of the network must equal the sum of the voltage increases around the circuit.

1. Find the various current in the circuit shown:

2. Find the various current in the circuit shown:

3. Find the row - echelon form of the augmented matrix of this system of linear equations :

$$\begin{aligned} x + y - z &= 3 \\ -x + 4y + 5z &= -2 \\ x + 6y + 3z &= 4 \end{aligned}$$

4. Carry of that augmented matrix (in the previous exercise) to reduced row - echelon form.

5. Find (if possible) conditions on a, b, c such that the system has no solution, one solution, or infinitely many solutions:

$$\begin{aligned} 3x + y - z &= a \\ x - y + 2z &= b \\ 5x + 3y - 4z &= c \end{aligned}$$